INTEGRALS OF NEWTONIAN EQUATIONS AND NONSTATIONARY ONE-DIMENSIONAL EXACT SOLUTIONS FOR MODELS OF MAGNETIZABLE FLUIDS WITH INTERNAL ANGULAR MOMENTUM

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1. Models of Magnetizable Liquids and Gases with Internal Angular Momentum

We shall assume that the models of magnetizable fluids correspond to a system of equations of the form

\[
\rho \frac{d}{dt} \left( \frac{v_a}{g} - \frac{h}{g} \epsilon \delta m \right) = -\partial_a p + \partial_b \left( \frac{Q_a b}{g} + \frac{h}{g} \epsilon \delta b m \partial_a v_b \right) +
\]

\[
+ \frac{1}{8\pi} (B_0 \partial_0 A^0 - H^a \partial_a B^0 + Di \delta_0 B^0 - E^a \partial_0 D^0) +
\]

\[
+ \frac{\rho}{c} \epsilon \delta m \frac{d}{dt} (E^a m^0) + \rho \epsilon \delta m (i^a + \rho \epsilon \delta m B^a) B^a;
\]

\[
\frac{d}{dt} m_a = g \epsilon \delta m H^0 \partial_a + \frac{1}{\rho} R_a; \quad \frac{\rho}{c} T + \frac{\partial A_0}{\partial S} = 0;
\]

\[
\rho T \frac{dS}{dt} = -\partial_a m^a = \epsilon \delta m \partial_a v^a + i^a \left( E^a + \frac{1}{c} \epsilon \delta m B^0 B^a \right) +
\]

\[
+ R_a \left( H^0 \partial_a + \frac{1}{2g} \epsilon \delta m \partial_0 \partial_a \right) - L^a \left( \frac{d}{dt} m_a + m_0 \partial_a v^a \right);
\]

\[
\frac{\partial}{\partial t} \rho + \partial_a \rho v^a = 0; \quad \frac{\partial}{\partial t} \rho_0 + \partial_a \rho_0 v^a = 0;
\]

\[
\partial_a D^a = 4\pi \rho_0; \quad \epsilon \delta m \partial_0 H^0 = \frac{1}{c} \frac{\partial}{\partial t} D^a + \frac{4\pi}{c} (\rho_0 \epsilon \delta m B^0).
\]

Here \( \partial_0 = \partial / \partial x^0 \) denotes partial differentiation with respect to the variables \( x^a \) (\( a = 1, 2, 3 \)) of the intertial Cartesian system of coordinates of the observer; \( \partial / \partial t \) denotes partial differentiation with respect to time \( t \); \( d / dt = \partial / \partial t + \nu^a \partial_a - \partial / \partial t \) denotes differentiation with respect to time following the motion of the fluid; \( E^a \) are the components of the velocity vector of individual points of the fluids; \( c \) is the free-space velocity of light; \( g \) is the gyromagnetic ratio; \( \epsilon \delta \) are the components of the unit Levi-Civita pseudotensor, antisymmetric with respect to all indices; \( \rho \) is the mass density of the fluid; \( \rho_0 = \rho_0 \) is the electric charge density of the fluid; \( \eta \) is a constant; \( S \) is the entropy per unit mass; \( T \) is the temperature in equilibrium processes; \( m_0 \) are the components of the magnetization vector of the fluid per unit mass within the framework of the nonrelativistic approximation, invariant with respect to the choice of the observer's inertial system; \( B^a \) are the components of the magnetic induction vector; \( H^a \) are the components of the magnetic field intensity vector; \( E^a \) are the components of the electric field intensity vector; \( D^a \) are the components of the electric induction vector. In accordance with the basic physical assumption adopted below, the electrical polarization of the fluid in the natural system of coordinates equals zero; accordingly, the following equalities hold for the components \( H^a \) and \( D^a \) in the observer's system:

\[
H^a = B^a - 4\pi \rho_0 m^a, \quad D^a = E^a + \frac{4\pi}{c} \rho \epsilon \delta m A_0.
\]

The components \( B^a \) and \( E^a \) can be determined through the scalar and vector potentials of the electromagnetic field:

\[ \{ E^a \} = - \text{grad} \Phi - \frac{1}{c} \frac{\partial}{\partial t} A, \quad \{ B^a \} = \text{rot} A. \] (3)

The components \( \tau_\alpha^\beta \) in Eqs. (1) determine the viscous stress tensor; \( L^\alpha \) and \( R^\alpha \) are generalized forces determining the processes of irreversible magnetization; \( i^\beta \) are the components of the conduction electric current vector; and \( q^\beta \) are the components of the heat-flux-density vector. For the components of the tensor \( Q_\alpha^\beta \) in (1) we have \( Q_\alpha^\beta = \tau_\alpha^\beta - m_\alpha L^\beta - (1/2g) \epsilon_\alpha^\beta \epsilon^\lambda_\lambda \). The components of all vectors and tensors introduced above are evaluated with respect to the observer's inertial system.

The quantities \( \tau_\alpha^\beta, i^\beta, q^\beta, R^\alpha, \) and \( L^\alpha \), which determine the dissipative processes in the fluid, must be prescribed as functions of the determinative parameters of the fluid and the field or be determined from additionally prescribed equations in such a manner that

\[
T\sigma = - \frac{1}{2} \frac{\partial \lambda_0}{\partial p} p\epsilon^{\alpha\beta} \frac{1}{c} \frac{\partial}{\partial \epsilon^{\alpha\beta}} \left( E^\beta + \frac{1}{c} \epsilon_{\beta\lambda\mu} B^\lambda - \frac{1}{2g} \epsilon_{\beta\lambda\mu} \epsilon^{\lambda\mu} R^\lambda \right) + \frac{1}{2g} \epsilon_{\beta\lambda\mu} \epsilon^{\lambda\mu} R^\lambda - \frac{1}{2g} \epsilon_{\alpha\beta} \epsilon^{\alpha\beta} R^\alpha - \frac{1}{2g} \epsilon_{\alpha\beta} \epsilon^{\alpha\beta} R^\alpha \geq 0; \tag{4}
\]

i.e., the internal production of entropy must be nonnegative.

The pressure \( p \) and the components \( H_\alpha^\lambda \) of the effective magnetic field vector in Eq. (1) are given by

\[
\begin{align*}
p &= - \rho \frac{\partial \lambda_0}{\partial p} \frac{1}{2} \rho m^\mu \left( \frac{1}{c} \epsilon_{\alpha\beta\lambda} \epsilon^{\alpha\beta} E^\lambda \right); \\
H_\alpha^\lambda &= B^\lambda - \frac{1}{c} \epsilon_{\alpha\beta\lambda} \epsilon^{\alpha\beta} E^\lambda + \frac{1}{2g} \frac{\partial \lambda_0}{\partial m^\mu} \left( \frac{1}{2g} \epsilon_{\alpha\beta\lambda} \epsilon^{\alpha\beta} E^\lambda \right) - \frac{2h}{g} \omega^\lambda,
\end{align*}
\]

where \( \lambda_0 \) is a given function of the arguments \( \rho, S, \) and \( m_\lambda \), and perhaps also of various constant tensors; \( \omega^\lambda = \frac{1}{2} \epsilon_{\alpha\beta} \epsilon_{\lambda}^{\alpha\beta} v_\lambda \) are the components of the vorticity vector; and \( h \) is a phenomenological constant.

The system of equations (1) incorporates the translational momentum equations, the continuity equation for both the mass density of the fluid and the electrical charge of the fluid, the angular momentum equation, Maxwell's equations for the field in the medium, the entropy balance equation, and the equation for the temperature. Equations (1) also give the energy equation

\[
\frac{\partial}{\partial t} \left[ \frac{1}{2} \rho v^2 - \frac{h}{g} \epsilon_{\alpha\lambda\mu} v_\alpha \partial_\mu m_\lambda + \rho U + \frac{1}{8\pi} \left( D_\alpha E^\alpha + B_\alpha H^\alpha \right) - \rho c \epsilon_{\alpha\lambda\mu} v_\alpha \epsilon^{\mu\lambda} m_\lambda \right] + \epsilon_{\alpha\beta\lambda} \epsilon^{\alpha\beta} E^\lambda + \frac{1}{2g} \frac{\partial \lambda_0}{\partial m^\mu} \left( \frac{1}{2g} \epsilon_{\alpha\beta\lambda} \epsilon^{\alpha\beta} E^\lambda \right) - \frac{2h}{g} \epsilon_{\alpha\beta} \epsilon^{\alpha\beta} v_\lambda = 0; \tag{6}
\]

and the heat-flux equation

\[
\begin{align*}
\frac{d}{dt} \left[ U + \frac{1}{2} m^\alpha \left( B_\alpha - \frac{1}{c} \epsilon_{\alpha\lambda\mu} v^\mu E^\lambda \right) \right] &= - \rho \left( \frac{1}{2} \rho + \frac{1}{2} \rho \left( B_\alpha - \frac{1}{c} \epsilon_{\alpha\lambda\mu} v^\mu E^\lambda \right) \right) \partial_\alpha v^\alpha + \left( B_\alpha - \frac{1}{c} \epsilon_{\alpha\lambda\mu} v^\mu E^\lambda \right) \times \frac{d}{dt} m_\alpha + \frac{1}{2} \rho \left( E_\alpha + \frac{1}{c} \epsilon_{\alpha\lambda\mu} v^\mu B^\lambda \right) - \frac{1}{2g} \partial_\alpha \left( \frac{h}{g} \epsilon_{\alpha\lambda\mu} m_\mu \partial_\lambda v_\lambda \right) + \frac{h}{g} \epsilon_{\alpha\lambda\mu} \epsilon^{\mu\lambda} \frac{d\sigma^\alpha}{dt}, \tag{7}
\end{align*}
\]

In Eqs. (6) and (7), the specific internal-energy density of the fluid described by Eqs. (1) is given by an equality of the form

\[
U = - \frac{1}{\rho} \lambda_0 (\rho, s, m_\alpha) - \frac{1}{2} m_\lambda \left( B_\lambda - \frac{1}{c} \epsilon_{\lambda\mu\nu} v_\mu E_\nu \right) + \frac{2h}{g} m^\mu \epsilon_{\mu\nu} \omega^\nu \quad \tag{8}
\]
The system of equations (1)-(7), with suitably prescribed $A_0$, $\tau_0$, $L^2$, $i\theta$, and $q^2$, can be used to describe, within the framework of Newtonian mechanics, viscous heat-conducting charged magnetized fluids (gases) with internal angular momentum. In the models under consideration, the internal angular momentum vector of the fluid per unit mass, defined by components $K^\alpha$, is proportional to the magnetization vector $gK^\alpha = m^2$. The momentum density vector of the medium, defined in the system $x^\alpha$ by components $I_\alpha$, is the sum of the momentum density vector connected with the motion of the mass and the internal momentum density vector connected with the presence of internal angular momentum:

$$I_\alpha = \rho v_\alpha = \frac{h}{g} a_{ab} \partial^\beta \rho m^\beta = \rho v_\alpha - h a_{ab} \partial^\beta \rho K^\beta.$$ 

2. Integrals of Differential Equations Describing Magnetizable Liquids and Gases

Let $\varphi(x^\alpha, t)$ and $\chi(x^\alpha, t)$ be fields of two-component spinors, prescribed by components $\varphi^A(x^\alpha, t)$ and $\chi^A(x^\alpha, t)$ ($A = 1, 2$) in a Cartesian system of coordinates with variables $x^\alpha$. We consider Eq. (1) for $h = 1$ and $k_0 = 0$, and set in them, by definition,

$$\rho = \rho^T \varphi + \chi^T \chi, \quad \rho^a = c(\rho^T \sigma^a \varphi - \chi^T \sigma^a \chi),$$

$$m^a = \mu S^a \gamma S^a, \quad \exp i(\xi(S) = \frac{\Omega + iN}{\sqrt{\Omega^2 + N^2}}).$$

Here $\eta(S)$ is a prescribed function with a nonzero derivative $\partial \eta/\partial S \neq 0$ in the entire range of variation of $S$; $m$ is a constant with the dimensions of specific magnetization; a dot above a symbol denotes the complex conjugate; the superscript $T$ denotes the transpose; and $\sigma^a$ are the Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

The quantities $\Omega$, $N$, and $S^a$ in (9) are given by

$$\Omega = \rho^T \chi + \chi^T \varphi, \quad N = i(\rho^T \chi - \chi^T \varphi), \quad S^a = -\rho^T \sigma^a \varphi - \chi^T \sigma^a \chi.$$ 

The symbols $\varphi$ and $\chi$ denote, respectively, columns of complex functions $\varphi^1, \varphi^2$ and $\chi^1, \chi^2$. By virtue of definitions (9), the components $m^a$ identically satisfy the equation

$$m_a m^a = m^2 = \text{const.}$$ 

We consider the following equations for the functions $\varphi(x^\alpha, t)$, $\chi(x^\alpha, t)$, $T(x^\alpha, t)$, $A_\alpha(x^\alpha, t)$ and $\Phi(x^\alpha, t)$:

$$\begin{align*}
\frac{d}{dt} \varphi - \sigma^a \partial_\alpha \varphi + i \partial_t \chi + i(-\tau_0 \sigma^a + \tau_1 \xi) \varphi &= 0; \\
\frac{d}{dt} \chi + \sigma^a \partial_\alpha \chi + i \partial_t \varphi + i(\eta_0 \sigma^a + \eta_1 \xi) \chi &= 0; \\
\rho T \frac{dS}{dt} &= -\partial_\theta \varphi + \tau_0 \partial_\theta \sigma^a + i \left( E_0 \frac{1}{c} \epsilon_{\beta \gamma} \rho^\beta B^\gamma \right) - \frac{B}{c} \left( \frac{d}{dt} m_\alpha + \mu_0 \partial_\alpha \varphi \right); \\
\partial_\alpha D^\alpha &= 4\pi q \rho; \quad \epsilon_{\alpha \beta \gamma} \partial_\beta \varphi \chi_\gamma = \frac{1}{c} \frac{d}{dt} D^\alpha + \frac{4\pi}{c} (i^a + q \rho \varphi).
\end{align*}$$ 

Here $I$ is the unit two-dimensional matrix, and the coefficients $\lambda$, $\tau$, and $\eta$ are given by

$$\lambda = \frac{g}{2mc(N+i\Omega)} \left[ -\frac{1}{\partial \eta/\partial S} \left( \frac{\partial A_\alpha}{\partial S} + \rho T \right) + \frac{mc}{g} \partial_\alpha S^a + \frac{m_0}{g c} \partial_\alpha (\nu_0 S^a) \right];$$

$$\eta^\alpha = \frac{1}{c} \frac{d}{dt} + \frac{g}{2mc} \left( -\frac{m}{g} \frac{\partial \eta}{\partial S} \frac{dS}{dt} + \frac{\partial A_\alpha}{\partial \rho} + B_0 m^2 \frac{1}{2} \rho v^2 - q \Phi + \frac{\delta A^*}{\delta \rho} \right);$$

$$\tau^\alpha = \frac{1}{c} \frac{d}{dt} + \frac{g}{2mc} \left( -\frac{m}{g} \frac{\partial \eta}{\partial S} \frac{dS}{dt} + \frac{\partial A_\alpha}{\partial \rho} + B_0 m^2 \frac{1}{2} \rho v^2 - q \Phi + \frac{\delta A^*}{\delta \rho} \right);$$

*Various models of magnetizable media have been considered previously in [1-13]. Equations (1)-(6), corresponding to a function $U$ defined by (8) for $h = 0$, are obtained in [10] using the variational principle.*
\[ \eta_a = \partial_a \gamma + \frac{g}{2mc} \left\{ -\frac{m}{\alpha \rho} \left( \delta \alpha - \frac{1}{m^2} m_{a} m_{a} \epsilon^0 \right) \left( \rho B_{a} - \frac{\rho}{c} \epsilon_{a \beta \gamma} \epsilon^0 \epsilon^\beta \epsilon^\gamma + \frac{\partial \Lambda_0}{\partial \mu_a} + L_a + \frac{\delta \Lambda^*}{\delta \mu_a} \right) + c \left( \frac{q}{c} A_a + Q_{a} + \frac{1}{c} \epsilon_{a \beta \gamma} \epsilon^0 \epsilon^\beta \epsilon^\gamma + v_a + \frac{\delta \Lambda^*}{\delta \eta_a} \right) - \frac{mc}{g} \frac{\partial \eta}{\partial \mu_a} \right\}; \]

\[ \tau_a = \partial_a \gamma + \frac{g}{2mc} \left\{ \frac{m}{\alpha \rho} \left( \delta \alpha - \frac{1}{m^2} m_{a} m_{a} \epsilon^0 \right) \left( \rho B_{a} - \frac{\rho}{c} \epsilon_{a \beta \gamma} \epsilon^0 \epsilon^\beta \epsilon^\gamma + \frac{\partial \Lambda_0}{\partial \mu_a} + L_a + \frac{\delta \Lambda^*}{\delta \mu_a} \right) + c \left( \frac{q}{c} A_a + Q_{a} + \frac{1}{c} \epsilon_{a \beta \gamma} \epsilon^0 \epsilon^\beta \epsilon^\gamma + v_a + \frac{\delta \Lambda^*}{\delta \eta_a} \right) + \frac{mc}{g} \frac{\partial \eta}{\partial \mu_a} \right\}. \]

Here \( \gamma(x^a, t) \) is an arbitrary differentiable function; \( \Lambda_0 \) is the function entering into the internal energy (8); and the symbols \( \delta \Lambda^*/\delta \rho \), \( \delta \Lambda^*/\delta \mu_a \), and \( \delta \Lambda^*/\delta \eta_a \) denote the Lagrange derivatives (variational derivatives) with respect to the corresponding arguments of \( \Lambda^* \). The quantity \( \Lambda^* \) in (13), of order \( c^2 \), is given by

\[ \Lambda^* = \frac{2\rho}{g} \left( -\frac{\alpha}{1 - \beta^2} - 1 \right) m_a \omega^a + \frac{2\rho \alpha}{g^2 (1 - \beta^2)} \left\{ -m_a \omega^a v^0 + \frac{1}{2} \epsilon_{a \beta \gamma} m_a \omega^a \frac{\partial}{\partial t} \left( \epsilon_{a \beta \gamma} \omega^a \right) \right\} \]  

For the coefficients \( \alpha \) and \( \beta \) in (13) and (14) we have

\[ \beta = \frac{|v|}{c}, \quad \alpha = \frac{1}{\rho} (S_\alpha S^\alpha)^{1/2} = \frac{1}{1 - \beta^2} \left[ 1 - \left( \frac{v_a m_a}{mc} \right)^2 \right]^{1/2}. \]

Clearly, within the framework of the nonrelativistic approximation, we can set \( \alpha = 1 \) and \( \Lambda^* = 0 \). The quantities \( Q \) and \( Q_a \) in the coefficients \( \lambda, \tau, \) and \( \eta \) are conventionally prescribed in such a manner that the equation

\[ \rho T \partial_a S + \frac{1}{c} \epsilon_{a \beta \gamma} \epsilon^0 \epsilon^\beta \epsilon^\gamma + \partial_\beta m_a = \rho \partial_a m_a + L_a \]

\[ + m_a \partial_\beta B^\beta = \rho \partial_\beta (\partial_\alpha Q_{a} - \partial_\alpha Q_{a}) + \partial_\alpha Q_{a} - \rho \frac{\partial}{\partial t} Q_{a} \]  

is satisfied identically or on the strength of Eq. (12). For arbitrary \( \tau_{a \beta}, i_{\beta}, \) and \( L^\beta \), Eq. (16) can serve as a definition of the functions \( Q \) and \( Q_a \).

The quantities \( \rho, m^a, v^a, \) and \( S \) in the coefficients \( \lambda, \tau, \) and \( \eta \) are assumed to be expressed in terms of \( \varphi \) and \( \eta \) in accordance with (9). In this manner, for prescribed functions \( Q \) and \( Q_a \), Eqs. (12) and (13) constitute a system of differential equations with which to determine the spinor fields \( \varphi(x^a, t) \) and \( \chi(x^a, t) \), the temperature \( T(x^a, t) \), and the scalar and vector potentials of the electromagnetic field \( A_a(x^a, t) \) and \( \Phi(x^a, t) \). It can be shown that all equations in (1) with \( R_a = 0 \) and \( h = 1 \) for arbitrary \( Q \), \( Q_a \), \( \tau_{a \beta}, i_{\beta}, \) and \( L^\beta \), satisfy Eq. (16), and that for an arbitrary function \( \gamma \) they are a consequence of Eqs. (12) and (13) if the mass density, entropy, velocity, and magnetization of the fluid are defined by relationships (9).

If the quantities \( Q \), \( Q_a \), and \( L^\beta \) in Eqs. (12) and (13) contain derivatives of the parameters \( \rho, m^a, v^a, \) and \( S \) of order not higher than the second, then the spinor equations in (12) are differential equations of the second order in \( \varphi \) and \( \chi \) in the general case and contain the components \( E_a \) and \( B_a \) without derivatives.

The system of hydrodynamic equations in (1), corresponding to the spinor equations in (12), on replacement of \( \rho, m^a, v^a, \) and \( S \) by \( \varphi \) and \( \chi \) in accordance with (9), becomes a system of equations of the third order in \( \varphi \) and \( \chi \) and of the first order in \( E_a \) and \( B_a \).

In this sense, Eqs. (12) and (13) can be regarded as integrals of Eqs. (1), \( \rho, m^a, v^a, \) and \( S \) being expressed in terms of \( \varphi \) and \( \chi \) through formulas (9).

It is not hard to see that the function \( \lambda^* \) and the terms in Eqs. (12) corresponding to it are quantities of order \( c^{-2} \). Accordingly, within the framework of the nonrelativistic approximation, terms in Eqs. (12) with the function \( \lambda^* \) can be neglected. Equations (12) with
coefficients \( \lambda, \tau, \) and \( \eta \) given by (13) without the terms with \( \Lambda^* \) are the exact equations for fluids described by a Lagrangian \( \Lambda' = \Lambda - \Lambda^* \), where \( \Lambda \) corresponds to Eqs. (1).

If we set \( q = 0 \) in Eqs. (12) and (13), which corresponds to the case of an uncharged fluid, then, in place of the scalar and vector \( \Lambda_0 \) and \( \Phi \), we can take the components \( E_a \) and \( B_a \) as the sought functions in (12) and (13). In this case, the system of equations (12) and (13) must be augmented by the second pair of Maxwell's equations:

\[
\varepsilon_{ab} \partial_t E_a = - \frac{1}{c} \frac{\partial}{\partial t} B_a, \quad \partial_a B^a = 0. \tag{17}
\]

We note that Eqs. (16) are satisfied identically on setting, for example,

\[
S = \text{const}, \quad Q = Q_0 = i^0 = L^0 = \tau_0 = 0 \tag{18}
\]
or if

\[
\frac{dS}{dt} = 0, \quad T = - \frac{d}{dt} A, \quad Q = A \frac{\partial S}{\partial t}, \quad Q_0 = A \partial_0 S, \quad \tau_0 = 0 = L_0 \tag{19}
\]

where \( A = A(x^a, t) \) is an arbitrary differentiable function. Clearly, these cases correspond to isentropic and adiabatic processes in ideal magnetizable fluids.

3. Exact Solutions for Models of Magnetizable Fluids with Internal Angular Momentum

Let us consider the isentropic motions of an ideal magnetizable uncharged fluid (gas) described by Eq. (1) with \( h = 1 \) and \( q = 0 \), in which the function \( \Lambda_0 \) depends only on the density \( \rho \), the entropy \( S \), and on the modulus \( \mu = \rho (m^a m^a)^{1/2} \) of the magnetization vector per unit volume:

\[
\Lambda_0 = \Lambda_0 (\rho, M, S). \tag{20}
\]

For the quantities \( i^0, \tau_0, \) and \( L^0 \) we take

\[
i^0 = 0, \quad \tau_0 = 0, \quad L_0 = 0. \tag{21}
\]

The pressure \( p \) and the components of the effective magnetic field \( H_a^\lambda \) are then given by expressions of the form

\[
p = -\rho \frac{\partial \Lambda_0}{\partial \rho} - M \frac{\partial \Lambda_0}{\partial M} - \frac{1}{2} \rho m^2 \left( B_a - \frac{1}{c} \varepsilon_{ab} v^a E^b \right);
\]

\[
H_a^\lambda = B_a - \frac{1}{c} \varepsilon_{ab} v_a E^b + \frac{\rho}{M} \frac{\partial \Lambda_0}{\partial M} m^b - \frac{2}{g} \omega^b. \tag{22}
\]

Calculation of the coefficients \( \lambda, \tau, \) and \( \eta \) in spinor equations (12), corresponding to the considered class of models of magnetizable fluids, gives, in the nonrelativistic approximation,

\[
\lambda = \frac{1}{2(N+i\Omega)} \left[ \partial_a S^a + \frac{1}{c^2} \frac{\partial}{\partial t} (v_0 S^a) \right];
\]

\[
\eta_\ell = \frac{1}{c} \frac{\partial \rho}{\partial t} + \frac{\rho}{2M} \left( B_a m^a - \frac{1}{2} \frac{v^2}{\rho} \frac{\partial \Lambda_0}{\partial \rho} + \frac{m^2}{2M} \frac{\partial \Lambda_0}{\partial M} \right); \tag{23}
\]

\[
\eta_\alpha = \frac{\partial \rho}{\partial t} + \frac{g}{2m} \left( v_a + \frac{1}{c} \varepsilon_{ab} m_b E^a \right) - \frac{g}{2c} \left( \delta_0^0 - \frac{1}{m^2} m_a m^a \right) B_\ell;
\]

\[
\tau_\alpha = \frac{\partial \rho}{\partial t} + \frac{g}{2m} \left( v_a + \frac{1}{c} \varepsilon_{ab} m_b E^a \right) + \frac{g}{2c} \left( \delta_0^0 - \frac{1}{m^2} m_a m^a \right) B_\ell. \tag{24}
\]

The quantities \( \Omega, N, \) and \( S^a \) in (21) are given by (10). Spinor equations (12) with coefficients (21) can be transformed in the nonrelativistic approximation to the simpler form

\[
\frac{1}{c} \frac{\partial \psi}{\partial t} = \sigma^a \partial_a \psi + \left\{ - \frac{g}{2m} \frac{\Omega - iN}{\rho} + \frac{1}{2(N+i\Omega)} \left[ \partial_a S^a + \frac{1}{c^2} \frac{\partial}{\partial t} (v_0 S^a) \right] \right\} \chi + i \left\{ \left( \partial_\alpha + \frac{g}{2c} B_\alpha + \frac{g}{2m} \varepsilon_{ab} m_b E^a \right) \sigma^a + \left( \frac{1}{c} \frac{\partial v_0}{\partial t} + \right. \right.
\]

\[
+ \]
\[
\begin{align*}
&+ \frac{g}{2mc} \left( c^2 - \frac{1}{2} v^2 + \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} \right) I \bigg\{ \varphi = 0; \bigg\}
&\frac{1}{c} \frac{\partial \chi}{\partial t} + \sigma \alpha \chi + \left\{ - \frac{g}{2m} \frac{\frac{c}{\rho} \Omega + iN}{1 + 2(N + i\Omega)} \left[ \frac{\partial_a S^a}{c} + \frac{1}{c^2} \frac{\partial}{\partial t} \left( v^a S_a \right) \right] \right\} \varphi +
&+ i \left\{ \left( \frac{\partial_a \gamma - \frac{g}{2c} B_a + \frac{g}{2mc} a_{ab} m^a E^a \right) \sigma^a + \left\{ \frac{1}{c} \frac{\partial \varphi}{\partial t} + \frac{g}{2mc} \left( c^2 - \frac{1}{2} v^2 + \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} \right) I \right\} \chi = 0. \right. \end{align*}
\]

Equations (22) differ from the spinor equations (12) with coefficients (21) by terms of order \( c^{-2} \).

We specify below some exact solutions of the system of equations (12), (19), and (21) for the isentropic case \( S = \text{const} \).

1. The system of equations (12), (19), and (21) permits an exact solution of the form
\[
E^a = \text{const}, \quad B^a = \frac{B}{m^a};
\]
\[
\varphi = \Phi_0 \exp \left\{ - \frac{g}{m} \left[ \left( v_a + \frac{g}{c} a_{ab} m^b E^b \right) \sigma^a + \left( B_a m^a - \frac{v^2}{2} + \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} \right) \right] \right\};
\]
\[
\chi = \chi_0 \exp \left\{ - \frac{g}{m} \left[ \left( v_a + \frac{g}{c} a_{ab} m^b E^b \right) \sigma^a + \left( B_a m^a - \frac{v^2}{2} + \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} \right) \right] \right\},
\]
where \( \Phi_0 \) and \( \chi_0 \) are arbitrary constant spinor fields, and \( B \) is an arbitrary constant. Quantities with the index "0" are evaluated for \( \varphi = \Phi_0 \), and \( \chi = \chi_0 \).

Solution (23) describes the translational motion of a uniformly magnetized fluid in a constant field. In particular, if we define \( \Phi_0 \) and \( \chi_0 \) by
\[
\varphi^1 = \chi^1 = 0, \quad \varphi^2 = \sqrt{\frac{1}{2} \rho_0} \exp \left\{ - \frac{i}{2} \eta(S_0) \right\}, \quad \chi^2 = \sqrt{\frac{1}{2} \rho_0} \exp \left\{ \frac{i}{2} \eta(S_0) \right\},
\]
then the fields \( \rho, v^a, m^a \), and \( S \) corresponding to solution (23) have the form
\[
\rho = \rho_0, \quad v^1 = v^2 = v^3 = 0, \quad m^1 = m^2 = 0, \quad m^3 = m, \quad S = S_0
\]
and describe a uniformly magnetized fluid at rest.

2. Let us consider nonstationary one-dimensional solutions of Eqs. (12), (17), and (21), dependent on time \( t \) and on the coordinate \( x^2 = z \), for which
\[
\varphi^1 = 0, \quad \chi^1 = 0, \quad B^1 = B^2 = 0, \quad E_1 = E_2 = E_3 = 0,
\]
\[
\varphi^2 = \sqrt{f_2} \exp ih_2, \quad \chi^2 = \sqrt{f_2} \exp ih_2.
\]
(24)

From formulas (9), we have the following for the components \( \rho, v^a, m^a, \) and \( S \) corresponding to fields \( \varphi \) and \( \chi \) given by (24):
\[
\rho = f_2 + h_2, \quad v^1 = v^2 = 0, \quad v^3 = \frac{f_2 - h_2}{f_2 + h_2},
\]
\[
m^1 = m^2 = 0, \quad m^3 = m, \quad h_4 - h_2 = \eta(S) = \text{const}.
\]
(25)

It follows from Maxwell's equations that the component \( B_a \) of the induction vector is constant for the class of solutions under consideration \( B_a = B = \text{const} \). While by (24) the condition that the entropy be constant is written in the form \( h_4 - h_2 = \text{const} \). Utilizing (24) and (25), the spinor equations (12) can be transformed to
\[
\left( \frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right) \frac{m}{\rho} \left[ c \left( \frac{f_2}{f_2 + h_2} \right)^2 - \frac{1}{4} c \left( \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} + mB \right) \right] ;
\]
\[-c \left( \frac{h_2}{\rho} \right)^2 + \frac{1}{4} c + \frac{1}{2c} \left( -\frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} + m B \right) \right] \right) h_4 = \frac{g}{m} \left( \frac{h_2}{\rho} \right)^2 + \frac{1}{4} c + \frac{1}{2c} \left( -\frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} + m B \right) \right) \right); \]  

(26)

\[
\left( \frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right)^2 h_2 = \left( \frac{\partial}{\partial z} - \frac{1}{c} \frac{\partial}{\partial t} \right)^2 h_4. \]  

(27)

Utilizing the constant-entropy condition, we eliminate the functions \( h_2 \) and \( h_4 \) from Eqs. (26) to obtain

\[
\left( \frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right)^2 \left[ c \left( \frac{h_2}{\rho} \right)^2 - \frac{1}{2c} \left( -\frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} \right) \right] + \]  

\[
\left( \frac{\partial}{\partial z} - \frac{1}{c} \frac{\partial}{\partial t} \right)^2 \left[ c \left( \frac{h_4}{\rho} \right)^2 - \frac{1}{2c} \left( -\frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} \right) \right] = 0. \]

Next, let us consider the case when the functions \( f_z \) and \( f_\rho \) depend only on the mass density: \( f_z = f_z(\rho) \) and \( f_\rho = f_\rho(\rho) \). In this case, the condition that Eqs. (27) be compatible gives

\[
\frac{df_z}{d\rho} \frac{d}{d\rho} \left[ c \left( \frac{f_z}{\rho} \right)^2 - \frac{1}{2c} \left( -\frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} \right) \right] + \]  

\[
+ \frac{df_\rho}{d\rho} \frac{d}{d\rho} \left[ c \left( \frac{f_\rho}{\rho} \right)^2 - \frac{1}{2c} \left( -\frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} \right) \right] = 0. \]  

(28)

Equation (28) can be expressed in the form

\[
\frac{df_z}{d\rho} \frac{d}{d\rho} \left( \frac{f_z}{\rho} \right)^2 + \frac{df_\rho}{d\rho} \frac{d}{d\rho} \left( \frac{f_\rho}{\rho} \right)^2 + \frac{\alpha^2}{2 \rho c^2} = 0. \]

(29)

where \( \alpha \) is given by

\[
\alpha = \left[ \rho \left( \frac{\partial \Lambda_0}{\partial \rho^2} + 2m \frac{\partial \Lambda_0}{\partial \rho \partial M} + m^2 \frac{\partial^2 \Lambda_0}{\partial M^2} \right) \right]^{\frac{1}{2}}. \]

(30)

Remembering from (25) that \( \rho = f_z + f_\rho \), for the general solution of Eq. (29) we find

\[
h_2 = \frac{1}{2} \rho \mp \frac{1}{2c} \rho \int \frac{a}{\rho} d\rho, \quad h_4 = \frac{1}{2} \rho \pm \frac{1}{2c} \rho \int \frac{a}{\rho} d\rho. \]

(31)

Here the arbitrary constant is included in the sign of the indefinite integral. This solution for \( f_z \) and \( f_\rho \) is used to determine, from Eqs. (26), the functions \( h_2 \) and \( h_4 \):

\[
h_2 = -\frac{1}{2} \eta(S) - \frac{g}{2m} \int \left[ v_y dz + \left( -\frac{1}{2} v^2 + \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} + m B \right) dt \right]; \]

\[
h_4 = \frac{1}{2} \eta(S) - \frac{g}{2m} \int \left[ v_y dz + \left( -\frac{1}{2} v^2 + \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} + m B \right) dt \right]. \]

(32)

By virtue of Eqs. (27), the integrals in (32) do not depend on the path of integration in the \((z, t)\) plane. In this manner, the spinor fields \( \Phi \) and \( \chi \) have the following form for the class of solutions under consideration:

\[
\Phi = \sqrt{\frac{1}{2} \rho \left( 1 \mp \int \frac{a}{\rho} d\rho \right)^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} \left( \eta(S) - \frac{g}{2m} \int \left[ v_y dz + \left( -\frac{1}{2} v^2 + \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} + m B \right) dt \right) \right\}; \]

\[
\chi = \sqrt{\frac{1}{2} \rho \left( 1 \mp \int \frac{a}{\rho} d\rho \right)^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2} \left( \eta(S) - \frac{g}{2m} \int \left[ v_y dz + \left( -\frac{1}{2} v^2 + \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} + m B \right) dt \right) \right\}; \]

\[
\eta = \int_0^t \left( \frac{g}{2m} \int v_y dz + \left( -\frac{1}{2} v^2 + \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} + m B \right) dt \right) \]
\[ \chi^2 = \sqrt{\frac{1}{2} \rho \left(1 \pm \int \frac{a}{c} \frac{dp}{\rho} \right)^n \exp \left( \frac{i}{2} \left\{ \eta(S) - \frac{g}{2m} \int \left[ v_3 dz + \left( -\frac{1}{2} v^2 + \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} + mB \right) dt \right] \right) \right) ; \]
\[ \varphi^1 = 0, \quad \varphi^2 = 0. \]

The dependence of the mass density \( \rho \) on the coordinate \( z \) and time \( t \) is found from the first equation in (27), which, utilizing expressions (31) for \( f_2 \) and \( f_4 \), can be brought to the form
\[ \frac{\partial}{\partial t} \rho + (v_3 \mp a) \frac{\partial}{\partial z} \rho = 0. \] (34)

We have from (34) that
\[ z = t(v_3 \mp a) + F(\rho), \] (35)

where \( F(\rho) \) is an arbitrary differentiable function of the density \( \rho \).

Relationships (31), (35), and the equality \( B = \text{const} \) completely define a series of exact solutions of (12), (19), and (21), for which
\[ v^3 = \mp \int \frac{a}{\rho} dp, \quad v^1 = v^2 = 0, \] (36)
\[ m^3 = m, \quad m^1 = m^2 = 0, \quad S = \text{const}. \]

Formulas (36) define solutions of Eqs. (1) and (20) in the form of simple waves. The quantity \( a \) defined by (30) has the dimensions of velocity and, in fact, is the velocity of propagation of the wave front. Utilizing expression (20) for the pressure \( p \), expression (30) can be brought to the form
\[ a = \left( \frac{\partial p}{\partial \rho} + m \frac{\partial p}{\partial m} + m^2 B \right)^n. \] (37)

In like manner to (24)-(33), it can be shown that the system of equations (12), (17), (18), and (21) permits the following exact solutions:
\[ \varphi^1 = \sqrt{\frac{1}{2} \rho \left(1 \mp \int \frac{a}{c} \frac{dp}{\rho} \right)^n \exp \left( \frac{i}{2} \left\{ \eta(S) - \frac{g}{2m} \int \left[ v_3 dz + \left( -\frac{1}{2} v^2 + \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} + mB \right) dt \right] \right) \right) ; \]
\[ \chi^1 = \sqrt{\frac{1}{2} \rho \left(1 \mp \int \frac{a}{c} \frac{dp}{\rho} \right)^n \exp \left( \frac{i}{2} \left\{ \eta(S) - \frac{g}{2m} \int \left[ v_3 dz + \left( -\frac{1}{2} v^2 + \frac{\partial \Lambda_0}{\partial \rho} + m \frac{\partial \Lambda_0}{\partial M} + mB \right) dt \right] \right) \right) ; \]
\[ \varphi^2 = \varphi^2 = 0, \quad B_3 = B = \text{const}, \quad B_1 = B_2 = 0, \quad E_1 = E_2 = E_3 = 0. \] (38)

Solution (38) of Eqs. (12), (17), (18), and (21) determines, through formulas (9), the solution of Eqs. (1), (17), (18), and (20) in the form of simple waves:
\[ v_3 = \mp \int \frac{a}{\rho} dp, \quad v_1 = v_2 = 0, \quad S = \text{const}, \] (39)
\[ m_3 = -m, \quad m_1 = m_2 = 0, \quad B_3 = \text{const}, \quad B_1 = B_2 = E_1 = E_2 = E_3 = 0. \]

In this case also, the dependence of the density \( \rho \) on the variables \( z \) and \( t \) is also given by (35).
LITERATURE CITED