VELOCITY STRUCTURE OF FLOWS IN NONUNIFORM CONSTANT MAGNETIC FIELDS

1. NUMERICAL CALCULATIONS

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Due to the finite dimensions of magnetic systems, situations always occur under the conditions of production and experiment in which the flow of a conducting fluid occurs in non-uniform magnetic fields. This results in an abrupt deformation of the field of the average velocities in the flow and requires consideration both in working out different designs of MHD machines and in producing MHD equipment for technological application.

The MHD flows which form in the zones of inflow into the interpolar space of a magnet and the outflow zones, and also in a magnetic field which varies periodically along the flow, were investigated in [1-6]. At the same time, a theoretical analysis was conducted on the condition that the magnetic field has only a single transverse component \( B_y(x) \). Concerning the second component, for example, \( B_z \), it was assumed that its contribution to the vorticity of the flow is negligibly small.

Using the example of a uniform flow in a rectangular tube in nonuniform fields of various configurations, it is demonstrated below that the analysis results depend significantly on whether or not the magnetic field has one or two components; the data from numerical calculation of each of the indicated cases can be modified, depending on the computational method selected.

The system of equations for a flow with velocity \( V = (u(y, z); 0; 0) \) in a channel of rectangular cross section \(-1/2 \leq y \leq 1/2, -b \leq z \leq b\), situated in a two-component magnetic field \( B = (0; B_y(y, z); B_z(y, z)) \), has the form

\[
\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + Ha^2 (J \times B) = C;
\]

\[
j = -\text{grad} \varphi + v \times B;
\]

\[
\text{div} j = 0
\]

in the induction-free approximation. Here \( Ha \) is the Hartmann number, \( j = (0; J_y; J_z) \) is the current density vector, and \( \varphi \) is the electric field potential. The unknown constant \( C = Re \varphi \partial / \partial z \) is determined from the integral condition specifying the flow rate

\[
\frac{b}{-b} \int_{-h}^{h} u(y, z) dy dz = 2b.
\]

Evidently, one can eliminate \( j \) from the system (1)-(3), after which we obtain, instead of (1) and (3),

\[
\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + Ha^2 \left[ B_y \left( \frac{\partial \varphi}{\partial z} - B_y u \right) - B_z \left( \frac{\partial \varphi}{\partial y} + B_z u \right) \right] = C;
\]

\[
\frac{\partial}{\partial y} \left( \frac{\partial \varphi}{\partial y} + B_y u \right) + \frac{\partial}{\partial z} \left( \frac{\partial \varphi}{\partial z} - B_z u \right) = 0.
\]

Taking account of the fact that the external magnetic field satisfies the condition \( \text{rot} B = 0 \), Eq. (6) can be converted to the form

\[
\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + B_z \frac{\partial u}{\partial y} - B_y \frac{\partial u}{\partial z} = 0.
\]

Let us assume that the component $B_y$ of the magnetic field is even and $B_z$ is odd along the $y$ axis (this case corresponds to the one shown in Fig. 4). Then the flow will be symmetric with respect to the plane $y = 0$, and if the channel walls are isolated, the boundary conditions are written in the form

$$
\begin{align*}
\frac{\partial u}{\partial z} &= 0 \quad \text{at } z = \pm b; \\
\frac{\partial u}{\partial y} &= 0 \quad \text{at } y = \frac{1}{2}; \\
\frac{\partial u}{\partial y} &= 0 \quad \text{at } y = 0.
\end{align*}
$$

If $B_y$ is also even along the $z$ axis, the flow still has the single plane of symmetry $z = 0$ on which $\partial u/\partial z = \varphi = 0$ (this case corresponds to the one shown in Fig. 2).

For the numerical calculation, Eqs. (5)–(7) were replaced by difference equations according to the following scheme.

Let $h = 1/2m$ and $l = b/n$ be the grid steps, where $m$ and $n$ are integral positive numbers. Let us introduce the grid operators

$$
\omega_{x,t} = \omega((i+s)h, (j+t)l);
$$

$$
\omega_{y,t} = \frac{1}{h}(\omega_{x,t+1} - \omega_{x,t}); \quad \omega_{z,t} = \frac{1}{l}(\omega_{x,t} - \omega_{x,t-1});
$$

and let us agree to omit the superscripts $s$ and $t$ at $s = t = 0$. On the basis of Eqs. (5) and (7), the difference scheme (let us call it scheme I) has the form

$$
\begin{align*}
u_{yy} + u_{zz} + \frac{h^2}{u_{y} + u_{y} + \frac{1}{2} \left[ B_x \left( \frac{\varphi_y + \varphi_y}{2} - B_y u \right) - B_z \left( \frac{\varphi_y + \varphi_y}{2} + B_z u \right) - C = 0, \\
\varphi_{yy} + \frac{B_z}{2} (u_{y} + u_{y}) - \frac{B_z}{2} (u_{y} + u_{y}) = 0.
\end{align*}
$$

Let us write another difference scheme for Eqs. (5) and (6), where $\varphi$ is sought on an offset grid (let us call it scheme II):

$$
\begin{align*}
u_{yy} + u_{zz} + \frac{h^2}{u_{y} + u_{y} + \frac{1}{2} \left[ B_x \left( \frac{\varphi_y - \varphi_y}{2} - B_y u \right) - B_z \left( \frac{\varphi_y - \varphi_y}{2} + B_z u \right) \right] - C = 0; \\
\varphi_{yy} + \frac{B_z}{2} (u_{y} + u_{y}) + \frac{1}{2} \left[ B_x \left( \frac{\varphi_y - \varphi_y}{2} + B_y u \right) \right] = 0.
\end{align*}
$$

Taking account of the experience of slow convergence of various kinds of iterative processes in the case of boundary conditions of the second kind, the exact solutions of the difference equations were found by the method of matrix factorization.

We will discuss two types of flows as characteristic examples. The first type is flow in a nonuniform symmetric magnetic field formed by pole pieces whose width is less than the channel width. The second type is flow in a nonuniform asymmetric magnetic field formed with the central axis of the channel coinciding with the face of the magnet poles. The induction distribution patterns in the magnetic systems which have been assumed in the calculations are illustrated in Fig. 1 and correspond to the relation

$$
B_y(y, z) + iB_z(y, z) = \frac{1}{1 + \left| y + i(z - z_0) \right|^2}.
$$

Here $i = \sqrt{-1}$, $z_0 = 0$ for a field which is symmetric with respect to general coordinate planes, and $z_0 = -b$ for a field which is symmetric with respect to $y = 0$. Equation (8) is selected so that the external field automatically satisfies the equations $\text{div } B = 0$ and $\text{rot } B = 0$. In addition, the magnetic field distributions obtained from (8) have satisfactory
agreement in the flow zone with the fields of actual magnetic systems.

The calculations were performed for \( b = 0 \) and 1.5 and Hartmann numbers \( \text{Ha} = 25 \) and 50 on three different grids with steps \( h = z = 0.1, 0.05, \) and 0.025. Let us consider the dependence of the data obtained on the computational scheme adopted in the different schemes.

The velocity and potential distributions are given in Fig. 2 for \( \text{Ha} = 50 \) in a flow situated in a nonuniform field which is symmetric with respect to \( y = 0 \) and \( z = 0 \). Curves 1, 2, and 3 correspond to the calculation based on scheme I with grid steps \( h = 0.1, 0.05, \) and 0.025, and curves 4 and 5 are based on scheme II with grid steps \( h = 0.1 \) and 0.025 (the results for \( h = 0.05 \) differ little from curve 5).

As is evident from the plots, the accuracy of the data obtained with the use of scheme I is very low, and it is necessary to further decrease the grid step in order to obtain reliable results. Reliable results are obtained already at a comparatively large step length when scheme II is used. In particular, the degree of deformation of the velocity distributions in the \( xz \) plane is similar in this case to what is observed experimentally [9], and zones of reverse flows are not observed, just as in [8] and [9]. The dependence of the results on the grid step is also significantly weaker.
Considering flows in nonuniform magnetic fields, one often performs the calculation with a single field component taken into account in order to simplify the problem, assuming that one can neglect the component $B_z$, since $B_z \ll B_y$ in actual situations \[7\]. At the same time, however, Eq. (7) is written for the potential $\Psi$, although it is evident that the condition $\text{rot} B = 0$ is not satisfied in this case. It is of interest to compare in similar situations the two computational alternatives with one- and two-component fields.

For the case under discussion, $B_z = 0$ and $B_y = B_y(0, z)$, taken from Fig. 1, correspond to a one-component field. The numerical calculations were done according to scheme I (scheme II is not suitable here), but the accuracy of the results obtained was, for a reason difficult to explain, incomparably higher than for a two-component field and the same computational difference scheme. Comparison of the data obtained is given in Fig. 3. Here the computational scheme with a one-component field is denoted by dashed lines and that with a two-component field by solid lines. As is evident from the plots, characteristic M-like velocity distributions are formed in the flow in a nonuniform symmetric magnetic field whose degree of deformation increases as the number $Ha$ increases. However, the run of the curves indicates a significant quantitative difference for the two computational alternatives indicated. When both field components are taken into account, appreciably less deformation of the velocity distribution occurs, and the zone of reverse flows near the flow axis is completely absent.

Only boundary conditions of the second kind are obtained for $\psi$ in the case of a magnetic field which is symmetric only with respect to the plane $y = 0$ ($z = -1.5$). In constructing difference schemes, one should always observe the following precaution: Not every difference scheme can give a consistent system of grid equations. Scheme II turns out to be suitable for this case. The numerical results for $Ha = 25$ and $Ha = 50$ are given in Fig. 4. The velocity and potential distributions are clearly asymmetric in such a channel. The fluid just flows around the zone with the maximum value of field induction. Just as in the previous case, the degree of deformation of the velocity distribution depends on the number $Ha$, but even at $Ha = 50$ a region of negative velocities is not observed, in contrast to \[7\].

Let us consider further two kinds of irregular flow in nonuniform constant magnetic fields: the flow in a field which varies periodically along the flow, in which the component of the gradient $\partial B_y / \partial z$ also occurs in addition to $\partial B_y / \partial x$, in contrast to \[5\], and the flow in a local magnetic field formed by circular pole pieces whose sizes are smaller than the flow boundaries.

The first kind of field can be represented in the form $B_y = B(z) \cos \pi x$ and the second as $B_y = B(r)$, where $r = \sqrt{x^2 + z^2}$. In both cases the field distributions obtained by means of an approximation to the experimental dependences for actual magnetic systems were adopted in the calculations. These dependences are given in \[9\].

In view of the complexity in the calculation of three dimensional problems, two-dimensional flow ($u_x, 0, u_z$) is considered, i.e., the finite dimensions of the channel along the field are not taken into consideration, and it is assumed that the magnetic field has only
one component \( B_y(x, z) \). Therefore, it is natural that the results of the analysis should be evaluated with a certain caution.

The numerical calculations were performed with the parameters \( \text{Re} = 20 \) and \( N = \frac{Ha^2}{\text{Re}} = 20 \) and 40 (one-half the channel height was taken as the characteristic size). In both cases, a system of equations for the stream function \( \psi \), vorticity \( \omega_y \), and potential \( \varphi \) of the following form was calculated:

\[
\frac{1}{\text{Re}} \nabla^2 \omega_y \frac{D(\psi, \omega_y)}{D(z, x)} + N \left[ \frac{\partial B_y}{\partial x} \left( \frac{\partial \varphi}{\partial x} + B_y \frac{\partial \psi}{\partial x} \right) + \frac{\partial B_y}{\partial z} \left( \frac{\partial \varphi}{\partial z} + B_y \frac{\partial \psi}{\partial z} \right) \right] = 0 ;
\]

\[
\nabla^2 \psi + \omega_y = 0 ; \quad \nabla^2 \varphi + B_y \omega_y = 0
\]

with the boundary conditions

\[
\begin{align*}
\omega_y |_{z=0} &= \psi = \varphi = 0 ; \\
\psi |_{x=1} &= -1 ; \quad \varphi |_{x=1} = 0 .
\end{align*}
\]

In addition, approximate Thom difference conditions of second-order accuracy were specified on the channel wall.

Additional periodicity conditions

\[
\begin{align*}
\psi |_{x_a} &= \psi |_{x_b} ; \\
\omega_y |_{x_a} &= \omega_y |_{x_b} ; \\
\varphi |_{x_a} &= \varphi |_{x_b} ;
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \psi}{\partial x} |_{x_a} &= \frac{\partial \psi}{\partial x} |_{x_b} ; \\
\frac{\partial \omega_y}{\partial x} |_{x_a} &= \frac{\partial \omega_y}{\partial x} |_{x_b} ; \\
\frac{\partial \varphi}{\partial x} |_{x_a} &= \frac{\partial \varphi}{\partial x} |_{x_b} ;
\end{align*}
\]

were specified in the calculation of the first kind of flow. In this case \( x_a = 0 \) and \( x_b = 2 \), since the complete period is equal to two.

Fig. 5. Flow pattern in a magnetic field varying periodically along the flow with an additional induction gradient \( \partial B_y / \partial z \); \( \text{Re} = 20 \) and \( N = 40 \).
In the calculation of the flow in a local field, conditions at the entrance and exit were specified instead of periodicity conditions. At the channel entrance (Poiseuille distribution)

\[ x = x_a; \quad \omega_y = -3z; \quad \psi = \frac{z}{2}(z^2 - 3); \quad \varphi = -B\psi. \]

At the channel exit

\[ x = x_b; \quad \frac{\partial \varphi}{\partial x} - \frac{\partial \omega_y}{\partial x} = \frac{\partial \psi}{\partial x} = 0. \]

For \( Re = 20, x_a = -1 \) and \( x_b = 3 \) were adopted.

For the numerical calculation the differential equations (9) were replaced by difference equations, which were calculated similarly to the equations discussed above.

The computational values obtained for the lines of constant vorticity \( \omega_y \), the streamlines \( \psi \), and the velocity distributions \( u_x \) are shown in Fig. 5 at various cross sections of a channel located in a periodic field with an induction gradient along the \( z \) axis. The curves on the plots indicate that the velocity distribution in such flows has a clearly expressed \( M \)-like velocity structure which varies periodically along the flow. We note that in contrast to the flow in a periodic field with \( \partial B_y/\partial z = 0 \) [5], the streamlines here are significantly distorted, and the degree of deformation of the velocity distribution is significantly higher; the reverse flows observed in individual zones are evidently due to inexactness of the calculation.

The lines of constant vorticity, streamlines, and velocity distributions are shown in Fig. 6 at various cross sections of a channel located in a local magnetic field (second type).

As is evident from the plots, \( M \)-like velocity distributions, which start to form already before the flow enters the region with maximum induction of the field, also originate in a flow in such a field. At the same time, the greatest deformation of the velocity structure is observed in the exit zone from the local field. Under the action of viscous forces, the velocity distribution is washed out and becomes continually closer to the Poiseuille distribution as the flow withdraws into the region in which the field induction decreases. The greatest vorticity of the streamlines and the maximum value of the vorticity are observed in the region in which the maximum field gradient occurs. We note that the qualitative pattern of the streamlines in such a flow is similar to the situation which arises in the case of the flow around objects when \( B = 0 \).

![Fig. 6](image)

**Fig. 6.** Flow pattern in a nonuniform magnetic field which is local with respect to the flow boundaries; \( Re = 20 \) and \( N = 40 \).
The results presented above for the numerical calculation of the flow of a conducting fluid in nonuniform magnetic fields indicate the possibility, in principle, of realizing steady-state velocity structures of a most unusual kind in the flow. However, the quantitative estimate of the velocity distribution parameters depends very significantly on the difference scheme selected, the accuracy of the calculation, and the nature of assumptions about the magnetic field distribution.

LITERATURE CITED