MATHEMATICAL MODEL OF CONDUCTING FLUID CONVECTION IN A NON-UNIFORM ALTERNATING MAGNETIC FIELD

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A mathematical model allowing to generate induction magnetic field, electric currents and Joule heat in a conducting cylinder placed in a non-uniform alternating magnetic field is proposed. The results of computations are given and analyzed.

Introduction. Technological progress in the aircraft industry is largely determined by the quality of construction materials. So a problem arises at the induction melting with the nickel super alloy technology related to the impurity distribution in the melt, which can be solved by the means of metal flow control in the melting bath before pouring the metal into the mould. Since the melting of refractory alloys occurs at temperatures of about 1800 K in a closed vacuum chamber, mathematical modelling is the most affordable way to study the basic laws of the process. There are significant temperature gradients in the melt at induction melting leading to intense convective motions. The mutual influence of magnetic fields and convective flows must be taken into account, which leads to the formulation of a complex magnetohydrodynamic problem. However, particular qualities of the technological process allow to divide the complete problem into sub-tasks: the calculation of magnetic field spatial distribution without considering the melt motion, the calculation of induction currents and volume heat source distribution, and the calculation of metal flows with a defined internal heat sources distribution. There are many researches on the melt interaction with an alternating electromagnetic field in ferrous [1] and non-ferrous metallurgy [2], including both experimental [3] and theoretical studies. The positive features of the magnetic field influence at metallurgical production described in the literature are the reduced time of the production cycle, grain refinement during solidification, homogenization of the chemical composition, and temperature field alignment. The aim of this work is the development of a mathematical model describing the magnetic field distribution, induction currents and Joule heat generation in the conducting melt. The modelled object is the induction melting process implemented at the precision casting workshop “Proton–Permskie Motory”, Perm, Russia.

1. Geometric and physical parameters. The furnace unit is shown schematically in Fig. 1. A nickel super alloy charge 1 is placed in a melting bath 2 sintered of a chamotte and corundum mixture. The melting bath is installed inside a water-cooled copper inductor coil 3 mounted on the asbestos base 4. The molten metal is poured through a corundum barrel nozzle 5. The induction furnace is installed inside a vacuum chamber. The melting process is conducted in technical vacuum (10^{-3} Pa). The technological parameters and the nickel melt physical properties [4] used for the simulation are listed in Table 1.

2. Governing equations. We consider a vertical cylinder filled with a paramagnetic \( \mu \approx 1 \) conducting melt exposed to a non-uniform alternating magnetic
field \( \mathbf{H}^{\text{out}} = \mathbf{H}(r, z) \cos \omega t \). The spatial magnetic field distribution has to be calculated, but at the current stage, it is assumed as defined. The dielectric walls of the mould do not affect the inductor magnetic field. Technical frequencies of the inductor coil are listed in Table 1. The alternating magnetic field generates electric currents in the melt, which represent volume heat sources. Since the heat sources are distributed irregularly in the liquid, a non-equilibrium temperature gradient arises and drives convective motions in the melt. The melt state governing equations contain the Maxwell equations, the Ohm’s law and the equations of thermogravitational convection in the Boussinesq approximation, the equation
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of motion contains the Lorentz force, and the heat equation considers Joule heat generation:

$$\text{rot } E = -\mu_0 \frac{\partial H}{\partial t},$$  \hspace{1cm} (1)

$$\text{rot } H = J,$$  \hspace{1cm} (2)

$$\text{div } E = \text{div } H = 0,$$  \hspace{1cm} (3)

$$\frac{\partial \psi}{\partial t} + (v \nabla) v = -\frac{1}{\rho} \nabla p + \nu \nabla^2 v + g \beta T \gamma + \frac{\mu_0}{\rho} J \times H,$$  \hspace{1cm} (4)

$$\text{div } v = 0,$$  \hspace{1cm} (5)

$$\frac{\partial T}{\partial t} + (v \nabla) T = a \nabla^2 T + \frac{J^2}{\rho c \sigma},$$  \hspace{1cm} (6)

where $H$ is the magnetic field, $t$ denotes time, $\mu$ denotes permeability, $\mu_0$ is the magnetic constant, $\sigma$ is the specific electric conductivity, $v$ is the velocity, $T$ is the temperature, $\gamma$ is the vertical unit vector, $a$ denotes thermal diffusivity, $J$ is the current density, $\rho$ denotes density, $c$ is the specific heat, $p$ is the pressure, $\nu$ is the kinematic viscosity, $g$ is the gravity acceleration, $\beta$ is the thermal expansion coefficient.

Let us analyze the relative influence of the magnetic field and melt convective flow on the estimates of some terms in the system of equations (1)–(6). We estimate the magnetic field penetration depth into the melt following [5]. Let there be a fixed melt exposed to an external field. Combining equations (1) and (2) yields

$$\frac{\partial H}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 H. \hspace{1cm} (7)$$

The order of magnitude estimation of the left- and right-hand parts of Eq. (7) gives the value of the characteristic magnetic field penetration depth into the melt $\delta \approx \frac{1}{\sqrt{1/\sigma \mu_0 \omega}} \approx 3 \cdot 10^{-3} \text{ m}$. We use the data from Table 1 for estimation.

The depth of magnetic field penetration into the melt is much less than the size of the melting bath, and we suggest that the magnetic field significantly affects the melt motion only within the Hartmann boundary layers adjacent to the vessel boundary [6].

To estimate the convection influence on the distribution of magnetic field and electric currents, we take both sides of the rotor of the Ohm’s law $J = \sigma E$ and use the electric field circulation theorem (1)

$$\text{rot } J = -\sigma \mu_0 \frac{\partial H}{\partial t} + \text{rot } v \times \mu_0 H. \hspace{1cm} (8)$$

With the inequality $\omega \gg v R^{-1}$ for the second term in the equation right-hand part, the influence of convective motion on the current distribution can be neglected.

Referring to the fact that in the Navier–Stokes equation the convective term $(v \nabla v)$ and the term with the lifting force $g \beta \Delta T$ have the same order of magnitude, we obtain $v \sim \sqrt{g \beta R \theta}$ for the characteristic velocity values, where the characteristic temperature difference $\theta = 20 \text{ K}$ and the flow are assumed steady. In this case, we get $v R^{-1} \approx R^{-1} \sqrt{g \beta R \theta} = 0.4 \text{ s}^{-1}$.

The frequency of the inductor coil current oscillations ranges $1 \div 2 \text{ kHz}$ and significantly exceeds the frequency critical value, at which the electric currents generated by the melt convective motion affect the magnetic field distribution.
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inside the melt. Thus, the conjugate magnetohydrodynamic problem can be considered as a consistent solution of the following sub-tasks: 1) the calculation of the spatial inductor magnetic field distribution in the melt without its motion; 2) the calculation of induction currents $J$ and volume power of Joule heat sources; 3) the solution of heat and mass transfer problem with a defined distribution of the magnetic field and internal heat sources.

3. Magnetic field and internal heat sources distribution. The solution of the magnetic field diffusion equation (7) in the metal is sought in the form of magnetic field components’ superposition: an inductor-created external harmonic $H^\text{out}$ and an internal $H^\text{ind}$ one created by the induction currents

$$H = H^\text{out} \cos \omega t + H^\text{ind}. \quad (9)$$

Inside the considered area rot $H^\text{out} = 0$.

The characteristic scales are the following: the inner radius of the melting bath $R_m \sim 0.1$ m for the coordinates; the inductor magnetic field inverse pulsating frequency $\omega^{-1} \sim 10^{-3}$ s for time; $H_0 = NI/4\pi R_m \sim 10^3$ A/m for the magnetic field value. Using Eq. (9), we derive the dimensionless equation for the magnetic field components

$$\frac{\partial H^\text{ind}}{\partial \tau} = \frac{1}{Rm} \nabla^2 H^\text{ind} + H^\text{out} \sin \tau, \quad (10)$$

$$Rm = \mu_0 \sigma R_m^2 \omega, \quad (11)$$

where $Rm$ is the magnetic Reynolds number. Substitution of the physical constants and process characteristic values (see Table 1) gives a value for $Rm \sim 10^3$.

Eq. (10) was solved in a cylindrical coordinate system. The solution was supposed to be independent of the azimuthal coordinate and symmetric about the $z$-axis; it made it possible to reduce the problem to a two-dimensional version and to perform calculations for half of the vertical cross-section of the cylinder.

The magnetic field is continuous at the conductor–dielectric boundary [5]

$$H_1 = H_2. \quad (12)$$

Here indices 1 and 2 denote different media.

The currents induced in the conductor are much less than those in the inductor coil, and their field decreases rapidly with distance form the conductor surface ($B^\text{ind}(\infty) = 0$), hence, we can neglect the magnetic field of the induced currents outside the conductor.

Fig. 2. Scheme to define the magnetic field of coils at the coordinates $(r_k, z_k, 0)$. 206
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As follows, the boundary condition for the magnetic field at the calculation area surface are

\[ r = 1, \quad z = 0, \quad z = h : \quad \mathbf{H} = \mathbf{H}^{\text{out}}. \quad (13) \]

The current density normal component is equal to zero at the symmetry axis

\[ r = 0 : \quad \mathbf{J} = \text{rot} \mathbf{H}^{\text{ind}} = 0. \quad (14) \]

The walls of the melting bath are non-conducting and diamagnetic at the melt temperature, therefore, from the magnetic field circulation theorem, they do not affect the strength of the coil magnetic field \( \mathbf{H}^{\text{out}} \).

The inductor magnetic field axial and radial components in the dimensionless form calculated using the Biot–Savart–Laplace law have the form

\[ \mathbf{H}_z = \sum_{k=1}^{N} \int_{0}^{2\pi} \frac{R - r_k \cos \phi}{(R^2 + r_k^2 + z_k^2 - 2Rr_k \cos \phi)^{3/2}} \, d\phi, \quad (15) \]

\[ \mathbf{H}_r = \sum_{k=1}^{N} \int_{0}^{2\pi} \frac{Rz_k \cos \phi}{(R^2 + r_k^2 + z_k^2 - 2Rr_k \cos \phi)^{3/2}} \, d\phi, \quad (16) \]

where \( R = 1 + \delta/R_m \) is the non-dimensional radius of the inductor coil, \( r_k \) and \( z_k \) are the radius-vector components from the element of a circuit with current of the

Fig. 3. Calculation results of the inductor magnetic field vector (a), the magnetic field radial component (b), the axial component (c), the resultant field in the melt (d), current density (e), and the internal heat source power (f).
inductor $k$-coil to the observation point, $\phi$ is the azimuthal angle of the cylindrical coordinates. Schematically, the calculation of formulas (15), (16) is illustrated in Fig. 2.

The problem (9)–(16) solving algorithm is implemented as a package of the Fortran program. We used an explicit finite difference scheme to approximate Eq. (10). Quasi-stationary solutions were derived by calculation until the amplitude of $\mathbf{H}^{\text{ind}}$ was established. The relative error of the induction magnetic field amplitude through the oscillation period does not exceed $\varepsilon = 10^{-5}$.

4. Numerical results. The results of numerical experiments are illustrated in Figs. 3,4. Figs. 3a,b,c show the solenoid magnetic field $\mathbf{H}^{\text{out}}$, the resulting field in the melt at $\text{Rm} = 1000$ (d), the azimuthal component of the induction current density (e) and the power of Joule heat sources (f). The conversion to

![Fig. 4](image)

Fig. 4. The distribution of the axial component of the resultant magnetic field along the radius (a) and height (b), the current density along the radius (c) and height (d), the volumetric capacity of heat sources along the radius (e) and height (f) for different values of Rm. 1 – Rm = 100, 2 – Rm = 1000, 3 – Rm = 10000. The boundary areas are shown in detail in the graphs; the inset show the distribution over the entire region at Rm = 1000.
the dimensional quantities $J$ and $qv$ is realized by multiplying by the dimensional factors

$$J_0 = \frac{H_0}{R_m} \sim 10^4 \frac{A}{m^2}, \quad q_0 = \frac{H_0^2}{\sigma R_m^2} \sim 500 \frac{W}{m^3}.$$  \hspace{1cm} (17)

As shown in Fig. 3d and Fig. 4a,b, the magnetic field is present only in the boundary area and is completely extinguished by the induction field in the central area. This is in good agreement with the estimates given above. There are electric currents in areas, where the magnetic field gradients are high, and at the end faces and at the side the currents have opposite directions (Fig. 3e and Figs. 4c,d). It should be noted that the currents at the side are significantly higher than those at the ends (Figs. 4c,d). The zone of intense heat generation is localized near the lateral surface, the heat at the end faces is much weaker (Fig. 4e,f). To establish regularities in the generation of heat, computations were performed with a different magnetic Reynolds number $R_m$, the results are shown in Fig. 4. It is found that the increasing magnetic Reynolds number provides a decrease of the magnetic field penetration depth into the conductor; the induction currents and the heat generation increase and localize closer to the surface.

**Conclusions.** To summarize, a generalized mathematical model, describing the magnetic field in a conducting cylinder placed in a non-uniform alternating magnetic field, has been developed. It allows to calculate the induction currents and generation of Joule heat. The distributions of the magnetic field, induction current density and volume heat source power were obtained by computer simulation methods. Based on the results of computations, regularities in variation of the control parameter (the magnetic Reynolds number) values have been found. This information later will allow to simulate the melt convective flow and reveal phenomena, which are important to understand the processes, which control the distribution of impurities.

**REFERENCES**


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