EFFECT OF MAGNETIC FIELD ON CONVECTION IN TWO-DIMENSIONAL HORIZONTAL
LAYER OF FLUID BOUNDED ABOVE BY A SOLID HEAT-CONDUCTING SLAB

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The stability of motion of a two-dimensional horizontal conducting layer of fluid in an external magnetic field has been studied in many papers, reviews of which are given in monographs [1] and [2]. In these papers the thermal conductivity of the solid walls bounding the fluid layer is usually taken into account on the assumption that the walls are infinitely thick. Also, allowing for the effect of a magnetic field leads to complex transcendental equations for the critical Rayleigh numbers, as a result of which an approximate variational method is utilized in [3] to determine the stability boundary. This method gives high accuracy for the instability growth level, but its error increases with increasing level number. For a nonconducting fluid, finite wall thickness is taken into account in an exact formulation in [3], where a horizontal layer of fluid is bounded below by a solid isothermal boundary and above by a solid heat-conducting slab of finite thickness. In the presence of a uniform external magnetic field, the approximate variational method in the inductionless approximation with Joule heating taken into account was utilized in [4] to study the instability ground (n = 1) level of a horizontal layer of conducting fluid bounded above and below by solid heat-conducting slabs of equal finite thicknesses and equal thermal conductivities.

In the present paper we investigate the boundary of monotonic instability of a two-dimensional horizontal layer of conducting fluid in an exact formulation. The horizontal layer of fluid occupies the region \( 0 \leq z \leq h, -\infty < x, y < +\infty \), where the \( z \) axis is directed upward. The boundary \( z = h \) is free, and on it heat transfer according to Newton's law takes place. The layer of fluid is in contact below with a solid heat-conducting layer of finite thickness occupying the region \( -l \leq z \leq 0, -\infty < x, y < +\infty \). The external magnetic field has the form \( B_0 = (0, 0, B_0) \). In the equilibrium state, the temperature of the fluid layer has a constant gradient \( (0, 0, -A) \). We introduce dimensionless quantities, taking \( h, h^2/\nu, A, \alpha/h, \) and \( h^{-\nu}(\rho \nu)^{1/2} \) as scales of length, time, temperature, velocity, and magnetic field. Here \( \nu \) is the kinematic viscosity, \( \alpha \) is the thermal diffusivity, \( \rho \) is the density, and \( \sigma \) is the electrical conductivity of the fluid. Also, \( Pr = \nu/\alpha \) and \( Pr_w = \nu/\alpha_w \) denote the Prandtl numbers, \( Pr_m = \nu/\nu_m \) is the magnetic Prandtl number, \( Ha = B_0h(\sigma/\rho \nu)^{1/2} \) is the Hartmann number, \( Ra = g\beta Ah^2/\nu a \) is the Rayleigh number, \( L = \alpha h/\lambda \) is the Biot criterion (\( \alpha_w \) is the thermal diffusivity of the wall, \( \nu_m \) is the magnetic viscosity, \( \beta \) is the coefficient of thermal expansion of the fluid, \( a \) is the heat-transfer coefficient, and \( \lambda \) is the thermal conductivity of the fluid). The \( z \) component of the velocity \( v_z(x, y, z, t) \) and of the induced magnetic field \( b_z(x, y, z, t) \), the temperature of the fluid \( T(x, y, z, t) \), and the temperature of the solid layer \( T_w(x, y, z, t) \) then satisfy the system of equations (see [2])

\[
\frac{\partial}{\partial t} \Delta v_z = \Delta \Delta v_z + Ra \Delta T + Ha \frac{\partial}{\partial z} \Delta b_z, \tag{1}
\]

\[
Pr \frac{\partial T}{\partial t} = \Delta T + v_z, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad \Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{2}
\]

\[
Pr_m \frac{\partial b_z}{\partial t} = \Delta b_z + Ha \frac{\partial v_z}{\partial z}, \tag{3}
\]

\[
Pr_w \frac{\partial T_w}{\partial t} = \Delta T_w. \tag{4}
\]

The boundary conditions are as follows:


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where \( \chi = \lambda_w / \lambda_0 \) and \( \lambda_w \) is the thermal conductivity of the solid layer. The boundary conditions for \( b_1 \) are not written down since they will not be utilized in the investigation of the boundary of monotonic instability.

We seek the solution of problem (1)-(5) in the form

\[
v_i = v(z) \exp [i(k_2 x + k_2 y) + p t], \quad T = \Theta(z) \exp [i(k_2 x + k_2 y) + p t],
\]

(6)

Inserting (6) into Eqs. (1)-(5), setting \( p = 0 \) (as we are investigating monotonic instability), and eliminating \( b(z) \), we obtain the system

\[
v'' - (2k^2 + iH \alpha) v' + k^2 v = k^2 \text{Ra} \Theta,
\]

(7)

\[
\Theta'' - k^2 \Theta = -v, \quad k^2 = k_2^2 + k_y^2, \quad \Theta_{\infty}'' - k^2 \Theta_{\infty} = 0
\]

(8), (9)

with boundary conditions

\[
at \ z = -1, \Theta_{\infty} = 0; \quad at \ z = 0, \Theta_{\infty} = \Theta, \quad \Theta' = \chi \Theta_{\infty}', \quad v = 0, \ v' = 0;
\]

(10)

\[
at \ z = 1, v = 0, \quad v' = 0, \quad \Theta_{\infty}' = -L \Theta.
\]

(11)

Boundary conditions (10) and (11) are inconvenient for application of a Fourier sine transform with respect to the variable \( z \). Accordingly, we consider, first of all, the auxiliary problem with boundary conditions

\[
at \ z = 0, v = 0, \quad v' = C_1, \quad \Theta = \Theta_{\infty} = C_2,
\]

(12)

\[
at \ z = 1, v = 0, \quad v'' = 0, \quad \Theta = C_3.
\]

(13)

where \( C_1 - C_3 \) are unknown constants.

Having solved Eqs. (7) and (8) subject to boundary conditions (12), we then find the unknown constants \( C_1 - C_3 \) from the third and fifth boundary conditions in (10) and the third boundary condition in (11).

The solution of problem (7)-(9), (12), (13) obtained by applying a finite Fourier sine transform has the form

\[
v(z) = \sum_{n=1}^{\infty} \frac{2n \chi \{ C_2 - (-1)^n C_3 - C_1 (n_2^2 + k^2) \} \sin n \pi z}{(n_2^2 + k^2)^3 + \text{Ra} n_2^2 (n_2^2 + k^2) - k^2 \text{Ra}},
\]

(14)

\[
\Theta(z) = \sum_{n=1}^{\infty} \frac{2n \chi \{ (n_2^2 + k^2)^2 + \text{Ra} n_2^2 (n_2^2 + k^2) \} C_2 - (-1)^n C_3 - C_1 \} \sin n \pi z}{(n_2^2 + k^2)^3 + \text{Ra} n_2^2 (n_2^2 + k^2) - k^2 \text{Ra}},
\]

(15)

\[
\Theta_{\infty}(z) = C_2 \frac{\sin k(z + 1)}{\sin k l}.
\]

(16)

On summing series (14) and (15), we obtain

\[
v(z) = k^2 \text{Ra} C_3 \sum_{i=1}^{3} \frac{A_i \sin n_i z}{\sin n_i} + \sum_{i=1}^{3} \frac{(k^2 \text{Ra} C_2 A_i - C_1 B_i) \sin n_i (1 - z)}{\sin n_i};
\]

(17)

\[
\Theta(z) = C_3 \sum_{i=1}^{3} \frac{D_i \sin n_i z}{\sin n_i} + \sum_{i=1}^{3} \frac{(D_i C_2 - A_i C_1) \sin n_i (1 - z)}{\sin n_i},
\]

(18)
where the coefficients $A_i$, $B_i$, $D_i$, and $n_i$ depend on the sign of the discriminant

$$D = \frac{1}{108} Ha^4 \left[ \frac{27k^4}{Ha^3} - Ha^4 k^4 - 4k^4 Ra Ha^2 - 18k^4 Ra - 4 Ha^2 k^6 \right]$$

of the cubic equation $(y + k^2)^3 + Ha^4 y (y + k^2) - k^2 Ra = 0$. If $D > 0$, then

$$A_i = \left[ 3r^2 (4 sh^2 q/3 + 3) \right]^{-1}, \quad r = \frac{1}{\sqrt{3}} Ha^3 k^3 + Ha^4 \text{sgn } q, \quad ch q = q/r^3,$$

$$q = (2 Ha^4 + 9 Ha^4 k^2 - 27 k^2 Ra)/54, \quad A_2 = a_2 - ia_3, \quad A_3 = a_2 + ia_3,$$

$$a_2 = -A_1/2, \quad a_3 = \frac{1}{2} \sqrt{3} A_1 \text{ cth } q/3, \quad B_1 = -A_1 (Ha^2/3 + 2r \text{ ch } q/3),$$

$$B_2 = b_2 - ib_3, \quad B_3 = b_2 + ib_3, \quad b_2 = A_1 (Ha^2/3 + 2r \text{ ch } q/3)/2,$$

$$b_3 = \sqrt{3} A_1 (s \text{ ch } q/3 + r \text{ sh } q/3)/(2 sh q/3), \quad s = -Ha^2/3 + r \text{ ch } q/3,$$

$$D_1 = A_1 (Ha^2/3 + 2r \text{ ch } q/3)^2 - Ha^4 (Ha^2/3 - k^2 - 2r \text{ ch } q/3), \quad D_2 = d_2 - id_3,$$

$$D_3 = d_2 + id_3, \quad d_2 = A_1 [3r (2s + Ha^2) \text{ ch } q/3 - s^2 + 3r \text{ sh } q/3 - Ha^2 (s - k^2)]/2,$$

$$d_3 = \sqrt{3} A_1 [s^2 - 3r \text{ sh } q/3 + Ha^2 (s - k^2) \text{ ch } q/3 + r (2s + Ha^2) \text{ sh } q/3)/(2 sh q/3),$$

$$n_1^2 = -Ha^2/3 - k^2 - 2r \text{ ch } q/3, \quad n_2^2 = -Ha^2/3 - k^2 + r \text{ ch } q/3 + i \sqrt{3} r \text{ sh } q/3,$$

$$n_3^2 = -Ha^2/3 - k^2 + r \text{ ch } q/3 - i \sqrt{3} r \text{ sh } q/3.$$

If $D \leq 0$, then

$$A_i = [(n_i^2 - n_{i+1}^2)(n_{i+1}^2 - n_{i+2}^2)]^{-1}, \quad n_4 = n_1, \quad n_5 = n_2,$$

$$B_i = A_i (n_i^2 + k^2), \quad D_i = A_i (n_i^2 + k^2)^2 + Ha^2 n_i^2, \quad n_1^2 = -2r \cos q/3 - k^2 - Ha^2/3,$$

$$n_2^2 = 2r \cos (60^\circ - q/3) - k^2 - Ha^2/3, \quad n_3^2 = 2r \cos (60^\circ + q/3) - k^2 - Ha^2/3,$$

$$\cos q = q/r^3.$$

Inserting (16)-(18) into the third and fifth conditions of (10) and the third condition of (11), we obtain a homogeneous system of equations for the unknown constants $C_1-C_3$:

$$C_1 S_1 - C_2 S_3 + C_5 (L + S_6) = 0, \quad C_1 S_2 - C_2 (S_3 + xk \text{ cth } kl) + C_5 S_5 = 0,$$

$$C_1 S_4 - C_2 k^2 Ra S_2 + C_5 k^2 Ra S_1 = 0,$$

where

$$S_i = \sum_{i=1}^{3} \frac{A_i n_i}{\sin n_i}, \quad S_2 = \sum_{i=1}^{3} \frac{A_i n_i}{\cos n_i}, \quad S_4 = \sum_{i=1}^{3} \frac{B_i n_i}{\log n_i},$$

$$S_5 = \sum_{i=1}^{3} \frac{D_i n_i}{\sin n_i}, \quad S_6 = \sum_{i=1}^{3} \frac{D_i n_i}{\log n_i}.$$

We note that $S_1$, $S_2$, and $S_4-S_6$ are real both for $D > 0$ and for $D \leq 0$. Equating the determinant of Eqs. (19) to zero gives a transcendental equation for the critical Rayleigh numbers:

$$\begin{vmatrix}
S_1 & S_5 & L + S_6 \\
S_2 & S_5 + xk \text{ cth } kl & S_6 \\
S_4 & k^2 Ra S_2 & k^2 Ra S_1
\end{vmatrix} = 0.$$

(20)
For \( \kappa = \infty \) (which corresponds to the condition \( \Theta = 0 \) at \( z = 0 \)), Eq. (20) ceases to depend on \( L \) and acquires the form

\[
k^2 Ra S_1^2 - (L+S_a)S_4 = 0. \tag{21}
\]

Finally, for \( \kappa = \infty \) and \( L = \infty \) (which corresponds to the condition \( \Theta = 0 \) at \( z = 0 \) and \( z = 1 \)) we obtain from (21)

\[
S_4 = 0. \tag{22}
\]

This case was studied by Chandrasekhar [1] by the approximate variational method. Equation (20) also ceases to depend on \( L \) when \( \kappa = 0 \) (case of a thermally insulating boundary \( z = 0 \)).

Figure 1 shows the results of a computer calculation [using expressions (20) and (21)] of the dependence of the minimum critical Rayleigh number \( Ra_m \) on the Hartmann number \( Ha \). The uppermost curve in Fig. 1 corresponds to the case considered by Chandrasekhar [1] by the approximate method. The results obtained in [1] and using formula (22) effectively coincide for the ground level [for example, for \( Ha = 5 \), formula (22) gives \( Ra_m = 1699.4 \) and \( k_m = 3.17 \), whereas in [1], \( Ra_m = 1699.4 \) and \( k_m = 3.17 \); for \( Ha = 50 \), formula (22) gives \( Ra_m = 35,043 \) and \( k_m = 6.75 \), whereas in [1], \( Ra_m = 35,044 \) and \( k_m = 6.75 \)]. The dependence of \( Ra_m(Ha) \) on \( \kappa \) for \( L = 1 \) and \( L = 0 \) is shown by curves 3-5. It can be seen from these curves that for small \( Ha \) the values of \( Ra_m \) increase sharply with increasing \( \kappa \) (from \( Ra_m = 335 \) for \( \kappa = 0 \) and \( Ha = 1 \) to \( Ra_m = 690 \) for \( \kappa = \infty \) and \( Ha = 1 \)). The effect of the Biot criterion \( L \) on \( Ra_m \) is shown by curves 2 and 4: with increasing \( L \), the values of \( Ra_m \) increase for all \( Ha \). The effect of the boundary conditions on the values of \( Ra_m \) diminishes with increasing \( Ha \) — all curves in Fig. 1 have the same asymptote as \( Ha \to \infty \).

Fig. 1. Dependence of minimum critical Rayleigh number \( Ra_m \) on the Hartmann number \( Ha \). 1) \( \kappa = \infty \), \( L = \infty \); 2) \( \kappa = 1 \), \( L = \infty \), \( L = 1 \); 3) \( \kappa = \infty \), \( L = 0 \); 4) \( \kappa = 1 \), \( L = 0 \), \( L = 1 \); 5) \( \kappa = 0 \), \( L = 0 \), \( L = 1 \).

Fig. 2. Dependence of the critical wave number \( k_m \) on the Hartmann number \( Ha \) for the same cases as in Fig. 1.

Fig. 3. Effect of the Hartmann number on dependence of \( Ra \) on \( k \) for \( \kappa = 1 \), \( L = 0 \), \( L = 1 \). 1) \( Ha = 7 \); 2) \( Ha = 5 \); 3) \( Ha = 3 \); 4) \( Ha = 0 \).
TABLE 1

<table>
<thead>
<tr>
<th>No.</th>
<th>Variants</th>
<th>$Ha=5$</th>
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<th>$Ha=50$</th>
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<td></td>
<td></td>
<td>$a_{2m}$</td>
<td>$R_{2m}$</td>
<td>$k_{2m}$</td>
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<td>13264</td>
<td>5.5</td>
</tr>
<tr>
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<td>$\alpha=1$, $L=\infty$, $L=1$</td>
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<td>14695</td>
<td>5.7</td>
</tr>
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<td>$\alpha=\infty$, $L=\infty$</td>
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<td>16428</td>
<td>6.1</td>
</tr>
</tbody>
</table>

Figure 2 shows the dependence of the minimum wave number $k_m$ on the Hartmann number, and Fig. 3 the effect of the Hartmann number on the dependence $Ra(k)$ for $\alpha = 1$, $L = 0$, and $L = 1$.

In contrast to the approximate method used in [1], formulas (20)-(22) yield, with one and the same accuracy, the critical Rayleigh number not only for the ground level ($n = 1$), but also for higher instability levels as well ($n = 2$, $3$, ...). Table 1 lists the values of the minimum Rayleigh numbers $Ra_{2m}$ and minimum wave numbers $k_{2m}$ for the second level ($n = 2$).

Formulas similar to (20)-(22) for various dispositions of a horizontal fluid layer and solid boundaries are readily derived using the method described in the present paper (a horizontal fluid layer bounded above and below by solid heat-conducting layers of finite thickness; two fluid layers separated by a solid heat-conducting layer and so on).

LITERATURE CITED