Various solar activity data have indicated that along with the well-known 22-year cycle there is a shorter periodicity of about 2 years. To simulate this phenomenon, we constructed a dynamical system, which reproduced double-periodic behaviour of the solar cycle. Such nonlinear dynamical system described the solar $\alpha\omega$-dynamo process with variable intensities $R_\alpha$ and $R_\omega$ of the $\alpha$-effect and the differential rotation, respectively. We have plotted the time distribution and butterfly diagrams for the poloidal and toroidal magnetic fields with dipole and quadrupole symmetries. The dynamical system with dipole symmetry of the magnetic field reproduces a regime similar to the double cycle at $-450 < R_\alpha R_\omega < -210$. In the case of quadrupole symmetry, this regime exists at $-220 < R_\alpha R_\omega < -190$.

Introduction. It is well known that the solar cycle has the main period, which is approximately equal to 22 years. Moreover, quasi-periodic 0.5–2-year oscillations have been detected in numerous studies based on the analysis of various solar signals and indices. These oscillations are named the “quasi-biennial oscillations”.

In [1], the data on solar magnetic fields since 1915 were inferred from H-alpha filament observations. On the basis of these data along with direct magnetographic observations quasi-periodic oscillations with a period of 1.3 years of the large-scale field were detected in the Sun during eight cycles. In [2], a source of poloidal magnetic field was registered using a uniform series of surface low-resolution magnetic field observations begun at the Wilcox Solar Observatory at Stanford. The results obtained confirm the idea that low-frequency dynamo waves with a period approximately equal to 22 years and a high-frequency wave of a quasi-two-year period can coexist. In [3], it is shown that the long-lived coronal structures are related to complexes of solar activity and display a quasi-periodic behaviour (in the form of impulses of coronal activity) with periods of 1.0–1.5 year, in the axisymmetric distribution of EUV and X-ray fluxes during the solar 23 year cycle. In [4], 1.3-year periodicities were found in the tracers of emerged flux at the solar surface, in particular, sunspots by the wavelet analysis.

In [5]–[10], oscillation periods from 1.0 to 2.0 years were isolated in various phenomena of solar and heliospheric activity. It was found that the rate and the amplitude of the oscillations differed from cycle to cycle. The 1.3–1.4-year periodicities are more frequently observed and better pronounced in the even cycles 18, 20, and 22, while the period of 1.5 years dominates in cycle 19, and that of 1.7 years in cycle 21.

In [11], the large-scale magnetic field data for 1960-2000 for several latitudinal belts were investigated using wavelet analysis, and the quasi-periodic $\sim 22$, 7- and 2-year components were selected. The main 22-year oscillation dominates in all latitudinal belts except the latitudes of $\pm 30^\circ$ from the equator. The butterfly diagram for the nominal 22-year oscillation shows a standing dipole wave in the low-latitude domain ($|\theta| \leq 30^\circ$) and another wave in the sub-polar domain ($|\theta| \geq 35^\circ$), which migrates slowly polewards. The nominal 7-year oscillation yields a
butterfly diagram with two domains. In the low-latitude domain ($|\theta| \leq 35^\circ$), the dipole wave propagates equatorwards and polewards in the sub-polar region. The nominal 2-year oscillation is much more chaotic than the other two modes, however, the waves propagate polewards whenever they can be isolated.

The quasi-biennial oscillations are mainly displayed in variations of the sector structure of the large-scale magnetic field. Thus, the quasi-biennial oscillations primarily represent variations of the equatorial dipole (and probably, to a lesser extent, of the quadrupole). They are clearly observed in the large-scale magnetic fields and locations of active longitudes, long-term dynamics of sunspot indices, geomagnetic activity, displacement of magnetic neutral lines, and in the heliospheric magnetic fields [12].

In [13], 233-year series of monthly means of relative sunspot numbers were analyzed for the 1749–1981 period by different statistical methods, such as autocorrelation power spectrum and complex demodulation analysis. It has been found that a quasi-biennial oscillation exists with a basic period of 25.6 months. The quasi-biennial oscillations start over with each solar cycle, thus giving approximately five consecutive two-year oscillations within one 11-year cycle. These oscillations within one solar cycle may not be coherent with those within the other cycle. In [14], a quasi-biennial fluctuation has been detected in the time-series of the sunspot umbra/penumbra area ratio on the basis of the Debrecen Photoheliographic Data. This study is based on an intermittent period of nearly eight years, the material comprises more than 18,000 individual sunspots.

Thus, the presence of the quasi-biennial oscillations on the background of the 22-year solar cycle is the fact well confirmed by observations.

For modelling the double magnetic solar cycle, we use the low-mode approach, which was proposed and developed in [15]–[20]. This approach is based on an assumption that the solar magnetic field can be described by non-linear dynamical systems with a relatively small number of parameters. Such non-linear dynamical systems are based on the equations of dynamo models. In the next section, we present two dynamical systems for the non-linear $\alpha \omega$-dynamo model.

Note that the theory of dynamical systems is widely used for modelling and studying phenomena that undergo spatial and temporal evolution. For example, the theory of dynamical systems is used to analyze the behaviour of complex atomic lattices, planetary motion, population dynamics, complex biological organisms. Dynamical systems with oscillations are useful to study the phenomena of multiple frequencies in physical systems of different nature. The theory of dynamical systems with oscillations is described in [21]–[23]. In [21], an application of dynamical systems is described for the study of nonlinear oscillations. In [23], oscillatory and wave processes in the systems of diversified physical natures, both periodic and chaotic, are considered from a unified point of view. The relation between the theory of oscillations and waves, nonlinear dynamics and synergetics is discussed.

1. The low-mode model. Generation and evolution of the solar magnetic field are described by a dynamo mechanism, the simplest version of which has been proposed in [24]. Parker dynamo equations are derived by averaging the magnetic field over the radius within the thin envelope, where the dynamo mechanism operates, and by neglecting the terms describing the effects of curvature near the pole. In this case, the dynamo equations have the form:

$$\frac{\partial A}{\partial t} = R_\alpha \alpha B + \frac{\partial^2 A}{\partial \theta^2}.$$  (1)
A double magnetic solar cycle and dynamical systems

\[ \frac{\partial B}{\partial t} = R_\omega \sin \theta \frac{\partial A}{\partial \theta} + \frac{\partial^2 B}{\partial \theta^2}. \]  

Here, \( B \) is the toroidal magnetic field, \( A \) is proportional to the toroidal component of the vector potential, which determines the poloidal magnetic field, and \( \theta \) is the latitude measured from the pole. The factor \( \sin \theta \) describes the decrease in length of a line of latitude near the pole. The second equation neglects the small contribution of the \( \alpha \)-effect, i.e. we use the so-called \( \alpha \omega \) approximation. Curvature effects are absent in the diffusion terms. It is assumed that the radial gradient of the angular velocity does not vary with \( \theta \). In Eqs. (1),(2), the parameters \( R_\alpha \) and \( R_\omega \) describe the intensity of the \( \alpha \)-effect and the differential rotation, respectively. We used a simple scheme to stabilize the magnetic field growth, namely, the algebraic quenching of helicity. This scheme assumes that \( \alpha = \alpha_0(\theta)/(1 + \xi^2 B^2) \approx \alpha_0(\theta)(1 - \xi^2 B^2) \), where \( \alpha_0(\theta) = \cos \theta \) is the helicity in a non-magnetized medium, and \( B_0 = \xi^{-1} \) is a magnetic field, for which the \( \alpha \)-effect is considerably suppressed.

The basic idea of the low-mode approach is that the mean-field dynamo equations are projected onto a minimum set of several first eigenfunctions of the problem describing the decay of magnetic fields without any generation sources. In this case, it is necessary to choose the minimum set of functions in such a way that the solution, which is a set of several first time-dependent Fourier coefficients taking into account the generation sources, will describe the general behaviour of the magnetic field of a given object and will not describe this field with any smaller set of functions. Substituting the chosen set of components of the magnetic field into the dynamo equations one can obtain a dynamical system of equations containing the selected modes.

In [20], the minimum set of first eigenfunctions was composed of the first two modes to describe the poloidal magnetic field and of the first two ones for the toroidal magnetic field with dipole symmetry. The obtained dynamical system reproduces the basic regimes (oscillation, vacillations, and dynamo bursts) observed in celestial bodies.

One can assume that adding other modes and using mixed dipole and quadrupole symmetries can lead to new regimes, for example, a regime, which is similar to the double cycle. So, we will represent dynamical systems with the minimum set of first eigenfunctions, which reproduce the double cycle of magnetic activity.

We found that a dynamical system with two additional modes and the dipole symmetry conditions \( A(0) = B(0) = A(\pi) = B(\pi) = 0 \) contains a regime, which is similar to the double cycle. In this case, the latitude distribution of the magnetic field takes the form:

\[ B(\theta, t) = b_1(t) \sin 2\theta + b_2(t) \sin 4\theta + b_3(t) \sin 6\theta, \]

\[ A(\theta, t) = a_1(t) \sin \theta + a_2(t) \sin 3\theta + a_3(t) \sin 5\theta. \]

Substituting the chosen set of components of the magnetic field into the dynamo equations and collecting the coefficients of the sines of similar arguments, we can obtain a dynamical system of six equations containing six modes:

\[ \dot{a}_1 = -a_1 + R_\alpha b_1/2 - 3/4 R_\alpha \xi^2 b_1^3 - 3/8 R_\alpha \xi^2 b_1^3 b_3 + 3/8 R_\alpha \xi^2 b_1^2 b_3 - 3/4 R_\alpha \xi^2 b_1 b_3 - (3) \]

\[ -3/8 R_\alpha \xi^2 b_1 b_3, \]

\[ \dot{a}_2 = 1/2 R_\alpha b_1 + 1/2 R_\alpha b_2 - 9 a_2 - 3/4 R_\alpha \xi^2 b_2^3 b_3 - 3/8 R_\alpha \xi^2 b_2^3 b_3 - 3/4 R_\alpha \xi^2 b_2 b_3 - (4) \]

\[ -3/8 R_\alpha \xi^2 b_1^3 - 3/4 R_\alpha \xi^2 b_1 b_2 b_3 - 3/8 R_\alpha \xi^2 b_1^2 b_3 - 3/4 R_\alpha \xi^2 b_1 b_3 - (4) \]

\[ -3/4 R_\alpha \xi^2 b_1 b_3, \]
Here, dots under \(a\) and \(b\) denote time derivatives.

In case of coexistence of dipole and quadrupole symmetries, the minimum set of first eigenfunctions to reproduce the double cycle contains ten modes:

\[
B(\theta, t) = b_1(t) \sin 2\theta + b_2(t) \sin 4\theta + b_3(t) \cos 2\theta + b_4(t) \cos 4\theta + b_5(t) \cos 6\theta,
\]

\[
A(\theta, t) = a_1(t) \sin \theta + a_2(t) \sin 3\theta + a_3(t) \cos \theta + a_4(t) \cos 3\theta + a_5(t) \cos 5\theta.
\]

The modes \(b_1(t) \sin \theta, b_2(t) \sin 4\theta, a_1(t) \sin \theta,\) and \(a_2(t) \sin 3\theta\) correspond to the field with dipole symmetry; \(b_3(t) \cos 2\theta, b_4(t) \cos 4\theta, b_5(t) \cos 6\theta, a_3(t) \cos \theta,\) \(a_4(t) \cos 3\theta,\) and \(a_5(t) \cos 5\theta\) correspond to the field with quadrupole symmetry. The dynamical system has the form:

\[
\dot{a}_1 = -a_1 + R_a b_1/2 - 3/4R_\alpha \xi^2(b_5(b_2b_4 - b_1b_3) + b_1(b_2^2 + b_3^2 + b_4^2 + b_5^2/2 + b_6^2/2)), \quad (8)
\]

\[
\dot{a}_2 = -9a_2 + R_a(b_1 + b_2)/2 - 3/4R_\alpha \xi^2((b_1^3 + b_2^3)/2 + b_1(b_2^2 + b_3^2/2 + b_4^2 + b_5^2)), \quad (9)
\]

\[
\dot{a}_3 = -a_3 + R_a b_3/2 - 3/4R_\alpha \xi^2(b_3^2/2 + b_4^2 + b_5^2/2 + b_6^2 + b_7^2 + b_9^2 + b_1^3 + b_2^3 + b_3^3 + b_4^3 + b_5^3 + b_6^3 + b_7^3 + b_8^3), \quad (10)
\]

\[
\dot{a}_4 = -9a_4 + R_a(b_3 + b_4)/2 - 3/4R_\alpha \xi^2((b_3^2 + b_4^2)/2 + b_3b_5^2 + b_4b_5^2 + b_3b_6^2 + b_4b_6^2 + b_3b_7^2 + b_4b_7^2 + b_3b_8^2 + b_4b_8^2 + b_3b_9^2 + b_4b_9^2), \quad (11)
\]

\[
\dot{a}_5 = -25a_5 + R_a(b_5 + b_6)/2 - 3/4R_\alpha \xi^2((b_5^2 + b_6^2)/2 + b_3b_5^2 + b_4b_6^2 + b_3b_7^2 + b_4b_8^2 + b_5b_6^2 + b_6b_7^2 + b_7b_8^2 + b_8b_9^2 + b_5b_9^2 + b_6b_9^2 + b_7b_9^2 + b_8b_9^2), \quad (12)
\]

\[
\dot{b}_1 = R_\omega a_1/2 - 3R_\omega a_2/2 - 4b_1, \quad \dot{b}_2 = 3R_\omega a_2/2 - 16b_2, \quad (13)
\]

\[
\dot{b}_3 = R_\omega a_3/2 - 3R_\omega a_4/2 - 4b_3, \quad \dot{b}_4 = 3R_\omega a_4/2 - 5R_\omega a_5/2 - 16b_4, \quad (14)
\]

\[
\dot{b}_5 = 5R_\omega a_5/2 - 36b_5. \quad (15)
\]
A double magnetic solar cycle and dynamical systems

Fig. 1. Higher modes of the toroidal magnetic field $B$ are functions of time. The upper plot shows $b_1$ (dash-dotted line), $b_2$ (solid line), and $b_3$ (dashed line) in case of dipole symmetry for $R_\alpha R_\omega = -360$. The lower plot shows $b_1$ (solid line) and $b_3$ (dashed line) in case of dipole and quadrupole symmetries for $R_\alpha R_\omega = -210$.

2. Toroidal and poloidal magnetic fields. The dynamical system (3)-(7) reproduces a regime similar to the double cycle at $-450 < R_\alpha R_\omega < -210$. In Fig. 1, higher modes of the toroidal magnetic field $B$ are presented as functions of time. The upper plot shows $b_1$ (dash-dotted line), $b_2$ (solid line), and $b_3$ (dashed line) in case of dipole symmetry for $R_\alpha R_\omega = -360$. At $-210 < R_\alpha R_\omega < -110$ the dynamical system has a regime of oscillations; with the further decrease of $R_\alpha R_\omega$ mixed oscillations appear.

In Fig. 2, latitude-time distributions (butterfly diagrams) of the magnetic field with dipole symmetry for $R_\alpha R_\omega = -360$ are shown. The upper plot shows the poloidal ($B_p = -\partial A/\partial \theta$) magnetic field and the lower one shows the toroidal field. In the plot for the toroidal field one can see the coexistence of short cycle against the background of longer patterns. The long patterns show the dipole wave propagating equatorwards, the short oscillations show the dipole wave prop-
agating polewards. The amount of short cycles against the background of a long cycle increases with the module of $R_\alpha R_\omega$. According [25], the large-scale surface magnetic field can naturally be considered as a tracer for the poloidal magnetic field. The plot for the poloidal field shows short oscillations, which correspond to dipole waves propagating polewards and to long patterns like standing waves. This behaviour of the poloidal magnetic field is similar to the observational results in [11] for superposition of smoothed butterfly diagrams for 18–22-year and 1.5–2.5-year spectral bands. However, the theoretical short patterns are regular unlike the observed patterns with a structure close to chaotic. Note that the dynamical system with dipole symmetry and the four modes represent the checkerboard structure of the butterfly diagrams like the structure of the smoothed butterfly diagrams for 18–22-year spectral bands in [11].

The dynamical system (8)–(15) reproduces a regime similar to the double cycle at $-220 < R_\alpha R_\omega < -190$. The lower plot of Fig. 1 shows the higher dipole mode $b_1$ with the solid line and the higher quadrupole mode $b_3$ with the dashed line in case of dipole and quadrupole symmetries for $R_\alpha R_\omega = -210$. At $-190 < R_\alpha R_\omega < -100$, the dynamical system has a regime of oscillations; with the further decrease of $R_\alpha R_\omega$ mixed oscillations appear.

Fig. 3 shows butterfly diagrams of the magnetic field with mixed dipole and quadrupole symmetries for $R_\alpha R_\omega = -210$. The upper plot shows the poloidal magnetic field and the lower one shows the toroidal field. In the plots for the toroidal

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*Fig. 2.* Latitude-time distributions of the magnetic field with dipole symmetry for $R_\alpha R_\omega = -360$. The upper plot shows the poloidal magnetic field, the lower one shows the toroidal field.
3. Summary. In Figs. 2, 3, one can see that the quasi-biennial oscillations are more pronounced for the toroidal magnetic field, while the observations show manifestations of the double cycle in the poloidal magnetic field. However, the dynamical systems generally reproduce the quasi-biennial oscillations for the solar magnetic field.

In case of \( R_\alpha R_\omega < 0 \), the dynamical system with six dipole modes has the following regimes: \(-110 < R_\alpha R_\omega < 0\) – decrease of the magnetic field magnitude; \(-210 < R_\alpha R_\omega < -110\) – oscillations; \(-450 < R_\alpha R_\omega < -210\) – mixed oscillations; \(-550 < R_\alpha R_\omega < -450\) – vascillations. For comparison, the dynamical system with four dipole modes has the following regimes: \(-100 < R_\alpha R_\omega < 0\) – decrease of the magnetic field magnitude; \(-220 < R_\alpha R_\omega < -100\) – oscillations; \(-340 < R_\alpha R_\omega < -220\) – growth of the magnetic field magnitude with the establishing of a stationary regime; \(-350 < R_\alpha R_\omega < -340\) – vascillations.
The dynamical system with six quadrupole modes exhibits a decrease in magnetic field magnitude at $-410 < R_\alpha R_\omega < 0$ and oscillations at $-416 < R_\alpha R_\omega < -410$. The regimes of the dynamical system with four dipole and six quadrupole modes are the following: $-100 < R_\alpha R_\omega < 0$ – decrease of the magnetic field magnitude; $-190 < R_\alpha R_\omega < -100$ – oscillations; $-220 < R_\alpha R_\omega < -190$ – mixed oscillations.

Thus, the obtained dynamical systems reproduce the regimes of a similar mixed cycle. The nature of the regimes depends on the parameters $R_\alpha$ and $R_\omega$.

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REFERENCES


A double magnetic solar cycle and dynamical systems


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