UDC 538.4:532.542.4

ON THE STABILIZING EFFECT OF A LONGITUDINAL MAGNETIC FIELD UPON THE NONHOMOGENEOUS TURBULENT MOTION OF A CONDUCTING FLUID

V. B. Levin

Magnitnaya Gidrodinamika, Vol. 1, No. 2, pp. 3-10, 1965

1. Turbulent shear flows can be approximately described using the equations of conservation of second-order moments of the fluctuation velocities and semiempirical approximations for the dissipative term and fluctuation energy transfer along the three coordinate axes [1, 2]. If a magnetic field is applied to the turbulent motion of an electrically conducting fluid, the Joule losses induced by the eddy currents will cause additional dissipation of fluctuation energy. A semiempirical formula for the approximate Joule dissipation was derived in [3]. Here we shall use the following expression for the dissipative term corresponding to the $\overline{u_i u_j}$ component of the Reynolds stress tensor:

$$\text{rate of dissipation} = \gamma \frac{B^2\sigma}{6\pi\rho} \overline{u_i u_j} \quad (\gamma \cong 1). \tag{1.1}$$

This is a nonrigorous extension to shear flows of the results obtained by Ievlev [4] for homogeneous flows.

The first term in Rotta's formula [1] for the dissipation of pulsating motion

$$2\nu \sum_{k=1}^{3} \overline{\frac{\partial u_i}{\partial x_k} \cdot \frac{\partial u_j}{\partial x_k}} = \delta_{ij} \frac{2c}{3} \frac{E^{3/2}}{l} + \nu c_1 \frac{\overline{u_i u_j}}{l^2} \tag{1.2}$$

corresponds to the dissipation associated with the energy flux distribution taken as a function of the scale spectrum; the second term represents the dissipation of all scale fluctuations (l is the "basic" scale) caused by direct viscosity effects. The second term on the right side of Eq. (1.2) can be combined with the Joule dissipation

$$\left(\frac{c_1\nu}{2l^2} + \frac{\gamma B^2\sigma}{6\pi\rho}\right)\overline{u_i u_j} = \frac{c_1\nu}{2l^2}\left(1 + \frac{\gamma\,\mathrm{Ha}_l^2}{3\pi c_1}\right)\overline{u_i u_j}. \tag{1.3}$$

where $\mathrm{Ha}_l = Bl\sqrt{\frac{\sigma}{\mu}}$ is the local Hartmann number.

Thus the effect of a magnetic field upon the fluctuations is equivalent to an increase of the coefficient c_1 by a factor of $\left(1 + \frac{\gamma \mathrm{Ha}_l^2}{3\pi c_1}\right)$. If the molecular viscosity did not enter into the equation of motion written for averaged flow quantities, it would be possible to speak about an increase of the effective viscosity of the fluid (as in the case of turbulent core flows or turbulent jets). It should be noted that the local Hartmann number changes across the flow together with the scale factor l. Thus the effective value of the coefficient c_1 varies over the section.

2. In the case of flows in pipes, ducts, and boundary layers with limited pressure gradients the diffusion and convection of fluctuation energy can be neglected within the limits of error of the present approximation, and the system of equations expressing the conservation of second-order moments can be written in the following form [2]:

$$\begin{aligned}
&\overline{uv}\frac{dU}{dy} + \frac{k}{2}\frac{\sqrt{E}}{l}\left(\overline{u^2} - \frac{2}{3}E\right) + \frac{c}{3}\frac{E^{3/2}}{l} + \frac{\nu c_1}{2l^2}\left(1 + \frac{\gamma\,\mathrm{Ha}_l^2}{3\pi c_1}\right)\overline{u^2} = 0; \\
&\frac{k}{2}\frac{\sqrt{E}}{l}\left(\overline{v^2} - \frac{2}{3}E\right) + \frac{c}{3}\frac{E^{3/2}}{l} + \frac{\nu c_1}{2l^2}\left(1 + \frac{\gamma\,\mathrm{Ha}_l^2}{3\pi c_1}\right)\overline{v^2} = 0; \\
&\frac{k}{2}\frac{\sqrt{E}}{l}\left(\overline{w^2} - \frac{2}{3}E\right) + \frac{c}{3}\frac{E^{3/2}}{l} + \frac{\nu c_1}{2l^2}\left(1 + \frac{\gamma\,\mathrm{Ha}_l^2}{3\pi c_1}\right)\overline{w^2} = 0; \\
&\overline{v^2}\frac{dU}{dy} + k\frac{\sqrt{E}}{l}\overline{uv} + \frac{\nu c_1}{l^2}\left(1 + \frac{\gamma\,\mathrm{Ha}_l^2}{3\pi c_1}\right)\overline{uv} = 0;
\end{aligned} \tag{2.1}$$

$$\overline{vw}\frac{dU}{dy}+k\frac{\sqrt{E}}{l}\overline{uw}+\frac{\nu c_1}{l^2}\left(1+\frac{\gamma \mathrm{Ha}_l^2}{3\pi c_1}\right)\overline{uw}=0;$$

$$k\frac{\sqrt{E}}{l}\overline{vw}+\frac{\nu c_1}{l^2}\left(1+\frac{\gamma \mathrm{Ha}_l^2}{3\pi c_1}\right)\overline{vw}=0,$$

(2.1) (cont'd)

where U is the mean velocity.

Solution of this system of equations gives the local relations, i. e., the dependence of the fluctuation characteristics at a given point upon the local Reynolds number $R_l=\frac{l^2 dU}{\nu dy}$. Obviously, all the local relations obtained in [2] for B = 0 are applicable here, if c_1 is replaced by $c_1\left(1+\frac{\gamma \mathrm{Ha}_l^2}{3\pi c_1}\right)$. The equation of motion written for the mean flow in a pipe or duct can be written in the following form:

$$\left(\frac{lv_*}{\nu}\right)^2\left(1-\frac{y}{a}\right)=\frac{R_E^2\left[cR_E+c_1\left(1+\frac{\gamma \mathrm{Ha}_l^2}{3\pi c_1}\right)\right]}{R_l}+R_l, \tag{2.2}$$

where $R_E=\sqrt{E}l/\nu$ (see [2]).

The right side of Eq. (2.2) can be rewritten in a more convenient form by means of the following notation:

$$F(R_l, \mathrm{Ha}_l^2)=\frac{R_E^2 cR_E+c_1\left(1+\frac{\gamma \mathrm{Ha}_l^2}{3\pi c_1}\right)}{R_l}+R_l \tag{2.3}$$

(see Fig. 1). Thus Eq. (2.2) becomes

$$\left(\frac{lv_*}{\nu}\right)^2\left(1-\frac{y}{a}\right)=F(R_l, \mathrm{Ha}_l^2). \tag{2.4}$$

The relations thus obtained permit one to determine the transverse distribution of the basic fluctuation parameters and the mean velocity profile, as Levin has done [2] for B = 0. It can be readily shown that in the transition region at the wall the two parameters R and Ha^2 can be replaced by a single universal parameter Ha^2/R_*^2 (see page 3). Indeed, if $y/a \ll 1$ and $lv_*/\nu = \varkappa\eta$, the relation (2.3) can be rewirtten in the following form:

$$\varkappa^2\eta^2=F(R_l, \mathrm{Ha}_l^2). \tag{2.5}$$

Substituting into Eq. (2.4) the obvious equality

$$\frac{\mathrm{Ha}_l^2}{\mathrm{Ha}^2}=\frac{l^2}{a^2}=\frac{\varkappa^2\eta^2}{R_*^2} \tag{2.6}$$

we obtain

$$\varkappa^2\eta^2=F\left(R_l, \varkappa^2\eta^2\frac{\mathrm{Ha}^2}{R_*^2}\right), \tag{2.7}$$

hence

$$R_l=R_l\left(\eta, \frac{\mathrm{Ha}^2}{R_*^2}\right). \tag{2.8}$$

All the remaining quantities can be expressed in terms of R_l (the universal velocity profile is shown in Fig. 2).

3. The critical relations derived in [2] can be rewritten for the case of flow in the presence of a magnetic field in the following form:

$$R_{EO} = \frac{c_1\left(\sqrt{1+\frac{8}{k/c}}-1\right)}{4c}\left(1+\frac{\gamma \mathrm{Ha}_l^2}{3\pi c_1}\right); \tag{3.1}$$

$$R_{lO} = c_1 f\left(\frac{k}{c}\right)\left(1+\frac{\gamma \mathrm{Ha}_l^2}{3\pi c_1}\right), \tag{3.2}$$

where

$$f\left(\frac{k}{c}\right) = \left[\frac{k}{4c}\left(\sqrt{1+8\frac{c}{k}}-1\right)+1\right]\sqrt{\frac{6\left[0.25\left(\sqrt{1+8\frac{c}{k}}-1\right)+1\right]}{\left(\frac{k}{c}-1\right)\left(\sqrt{1+8\frac{c}{k}}-1\right)}}. \tag{3.3}$$

Let us examine separately the critical conditions applicable to a turbulent core flow and to viscous flow at the wall.

a) A turbulent core flow is characterized by large values of the local Reynolds number and, consequently, by large critical values of the local Hartmann number $\frac{\mathrm{Ha}_l^2}{3\pi c_1} \gg 1$. Hence Eqs. (3.1) and (3.2) reduce to

$$R_{EO} = \frac{\sqrt{1+\frac{8}{k/c}}-1}{12\pi c}\mathrm{Ha}_l^2; \tag{3.4}$$

$$R_{lO} = \frac{1}{3\pi} f\left(\frac{k}{c}\right)\mathrm{Ha}_l^2. \tag{3.5}$$

Equation (3.5) yields an expression for the transverse distribution of critical values of the local Reynolds number

$$R_{lO} = \frac{1}{3\pi} f\left(\frac{k}{c}\right)\mathrm{Ha}^2\frac{l^2}{a^2}, \tag{3.6}$$

and thus an expression for the critical values of the first derivative of the mean velocity

$$\left(\frac{dU}{dy}\right)_0 = \frac{1}{3\pi} f\left(\frac{k}{c}\right)\frac{B^2\sigma}{\rho}. \tag{3.7}$$

Let us now define a Reynolds number $R_* = v_* a/\nu$, where v_* is the dynamic viscosity, and a is the radius of the pipe (or half-span of the duct). We rewrite Eq. (2.4) in the following form:

$$R_*^2\frac{l^2}{a^2}\left(1-\frac{y}{a}\right) = F(R_l,\ \mathrm{Ha}_l^2). \tag{3.8}$$

Consider flow at a fixed Hartmann number. To each value of R_* by Eq. (3.8) corresponds a transverse R_l distribution. At sufficiently high values of R_* the magnetic field effect can be neglected and the R_l distribution has a characteristic maximum at half-radius. With decreasing R_* values the shape of the curve changes, and it approaches its critical form (see Fig. 3). Note that over a large interval of R_* values the R_l distribution is not much different from the critical distribution. This can be attributed to the sharp decrease of turbulent viscosity in the neighborhood of the local critical regime, and indicates a tendency toward simultaneous onset of the critical regime over the entire cross section (see [2]).

b) For a description of the viscous region at the wall we shall use Eq. (2.7) which, when R_{l0} is expressed in terms of Ha^2, has the following form:

$$\varkappa^2\eta^2 = F_0\left(\varkappa^2\eta^2\frac{\mathrm{Ha}^2}{R_*^2}\right). \tag{3.9}$$

Equation (3.9) defines a universal (i.e., valid for any Hartmann number) distribution of the local critical values of R_{*0} in the viscous region (see Fig. 4):

$$\frac{R_{*0}}{Ha}=\Phi(\eta). \tag{3.10}$$

c) An analysis of the results presented in Fig. 3 leads to the conclusion that at some critical value of the Reynolds number R_{*0} a critical regime sets in instantaneously over the entire cross section. Obviously, this critical value must

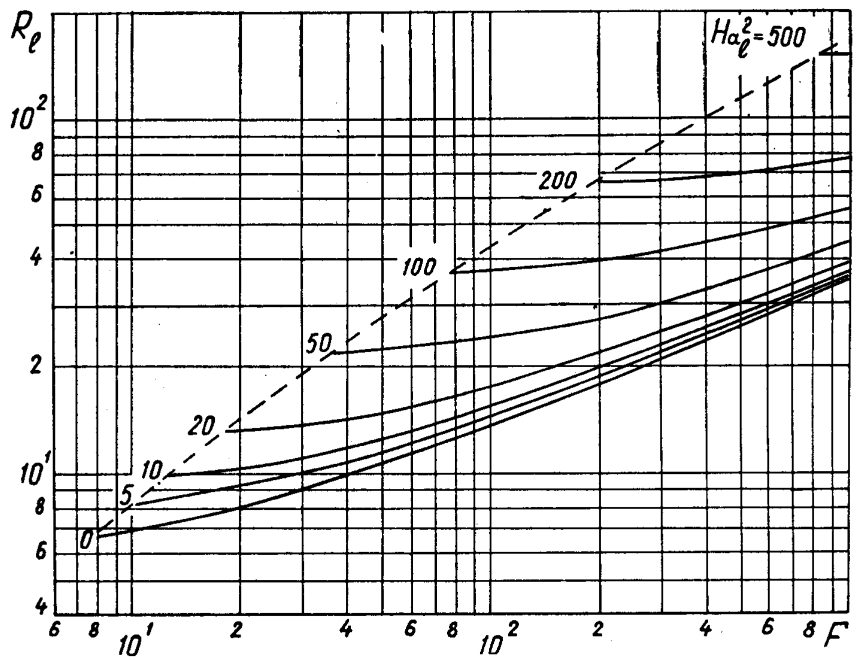

Fig. 1. Relation between R_l and the function F in accordance with Eq. (2.3).

be larger than the minimum critical value presented in Fig. 4.

Thus the minima

$$R_{*0}=0.83\,Ha, \quad \eta_0=16.6 \quad \left(\text{for } \frac{k}{c}=7\right),$$
$$R_{*0}=1.1\,Ha, \quad \eta_0=16.6 \quad \left(\text{for } \frac{k}{c}=1.6\right), \tag{3.11}$$

established the lower limit of the critical Reynolds number. In calculating the average flow velocity associated with the critical regime, starting with some value of R, one may assume that $U_\omega = U_l$ at the wall (subscript "l" denotes the outer boundary of the laminar sublayer) and calculate $(dU/dy)_0$ in the turbulent core flow in accordance with Eq. (3.7). Thus, it follows that

$$R_0=\frac{U_{cp}2a}{\nu}=\frac{U_л 2a}{\nu}+\frac{1}{3}\left(\frac{dU}{dy}\right)_0\frac{2a^2}{\nu}=2\eta_0 R_{*0}+\frac{2}{9\pi}f\left(\frac{k}{c}\right)Ha^2. \tag{3.12}$$

Using Eq. (3.11), we get

$$R_0=28\,Ha+0.2\,Ha^2 \quad \left(\text{for } \frac{k}{c}=7\right);$$
$$R_0=36\,Ha+0.33\,Ha^2 \quad \left(\text{for } \frac{k}{c}=1,6\right). \tag{3.13}$$

The estimated lower limit of applicability of Eq. (3.13) is around $R_0 \approx 5000$. The critical curve is given in Fig. 5.

As can be seen, the quantities measured along the coordinate axes include the radius of the pipe as a factor. Thus the change of radius is represented by a straight line passing through the center of the coordinate system. Consider the

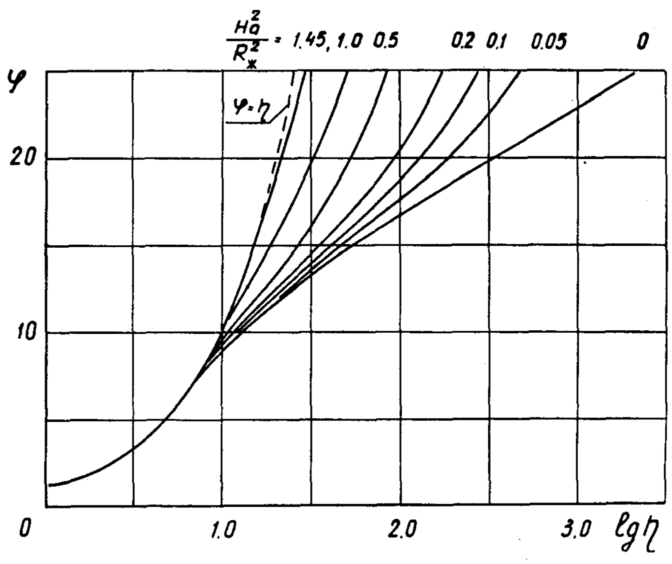

Fig. 2. Universal velocity profile in the transition region at the wall.

value of R at which the straight line becomes tangent to the critical curve. In the region to the left of this R value a decrease in radius stabilizes the flow (the representative point moves from the critical to the laminar flow regime). On the right of this R value a decrease in radius reduces the stability of the flow (the representative point moves from the critical to the turbulent regime).

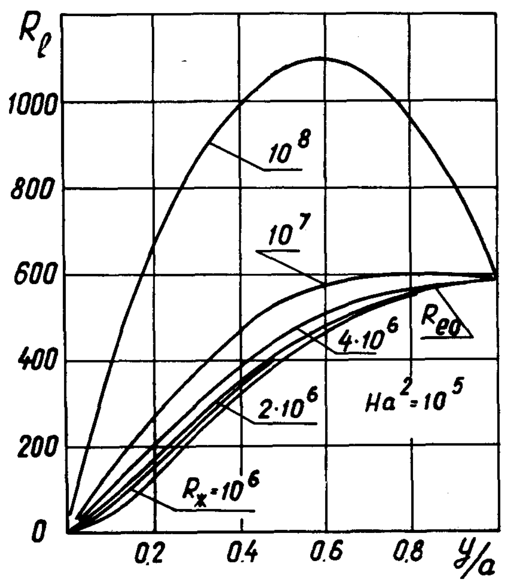

Fig. 3. Distribution of local Reynolds number R_l for various values of R_*.

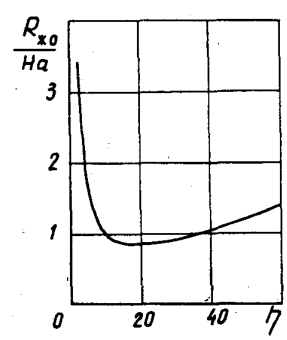

Fig. 4. Distribution of local critical values of R_{*0} in the viscous region.

4. Let us estimate the stabilizing effect of a longitudinal magnetic field on jet-type flows. A comparison with conventional turbulent jets indicates that the turbulent viscosity and the expansion of the jet are significantly reduced in the near-critical regime. Thus, in the critical regime the diffusion of the fluctuation energy can be neglected with the same approximation that has been applied to boundary layer flows. However, in contrast with boundary layer flows, where molecular viscosity is important at the wall, in jet flows this viscosity becomes negligible. Furthermore, the scale of turbulence may be considered constant across the jet. Under these conditions, and in accordance with the results of § 2, the effect of the longitudinal magnetic field is equivalent to an increase in molecular viscosity by a factor of $\left(1+\frac{\gamma \mathrm{Ha}_l^2}{3\pi c_1}\right)$

$$\mu_e = \mu\left(1 + \frac{\gamma \mathrm{Ha}_l^2}{3\pi c_1}\right). \tag{4.1}$$

The molecular viscosity μ_e is constant in the transverse direction; for moderate values of the Reynolds number it increases in the direction of flow as the square of the scale factor. At some cross section the Reynolds number based on the effective viscosity falls to its critical value. Downstream of this section the turbulence attenuates. The critical value of the Reynolds number corresponding to the presence of a magnetic field R_0 is related to the critical value corresponding to zero magnetic field R_{00} by the following expression:

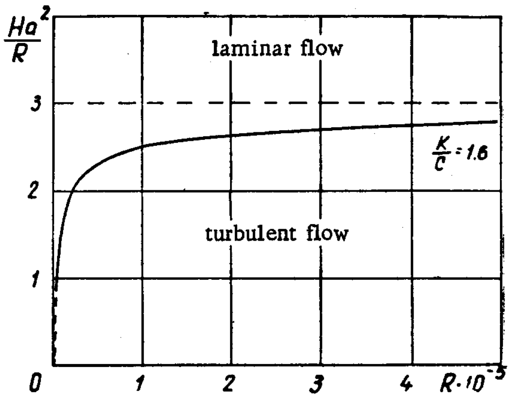

Fig. 5. Critical curve for flow of an electrically conducting fluid in the presence of a longitudinal magnetic field.

$$R_0 = R_{00}\left(1 + \frac{\gamma \mathrm{Ha}_l^2}{3\pi c_1}\right). \tag{4.2}$$

5. The results thus obtained provide a detailed qualitative description of the above phenomena. The corresponding quantitative relations can be used for estimating purposes, the design of experiments, and the analysis of experimental results. Experiments at large Hartmann numbers require the application of strong magnetic fields to large volumes; therefore the possibility of extrapolation of the experimental results by means of a simple functional relation between R_0 and Ha, as given in Eq. (3.13), is quite advantageous. In particular, an extrapolation to large Reynolds numbers (up to the self-similar region) permits one to extend results obtained for duct flows to jet-type flows.

REFERENCES

1. J. Rotta, Ztschr. f. Phys., 129, 547, 1951.
2. V. B. Levin, Teplofizika Vysokikh temperatur, 4, 588, 1964.
3. D. S. Kovner and V. B. Lenin, Teplofizika Vysokikh temperatur, 5, 742, 1964.
4. V. M. Ievlev, Proceedings of the 4th Riga Conference on Magnetohydrodynamics [in Russian], Riga, 1964.

3 September 1964