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TWO-DIMENSIONAL MAGNETOHYDRODYNAMIC JETS

A. B. Tsinober and E. V. Shcherbinin

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Similar solutions of magnetohydrodynamic boundary layers, the solutions of jet flows in particular, are usually restricted to configurations where the magnetic field is a power function of the longitudinal coordinate x. Solutions corresponding to transverse magnetic fields expressible as arbitrary functions of the x coordinate are presented.

1. Unbounded Submerged Jet of Electrically Conducting Fluid

We seek a solution to this problem under the following conditions: the fluid is assumed to be incompressible, its electrical conductivity σ is constant over the entire flow region. The applied magnetic field **B** is perpendicular to the axis of the jet, its perturbation by the flow is neglected, i.e., the formulation of the problem corresponds to the usual small magnetic Reynolds number $Re_m \ll 1$ approximation. Assuming, furthermore, that the electric currents vanish outside the mixing region, we obtain the following set of equations:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2}{\rho}u, \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2}$$

$$v=\frac{\partial u}{\partial y}=0 \quad \text{at} \quad y=0, \quad u\to 0 \quad \text{at} \quad y\to\pm\infty. \tag{3}$$

The boundary conditions given in (3) are supplemented by the integral relation obtained by integrating Eq. (1) over the jet cross section

$$\frac{d}{dx}\int_{-\infty}^{+\infty}u^2 dy=-N\int_{-\infty}^{+\infty}u\,dy, \tag{4}$$

where

$$N(x)=\sigma B^2(x)/\rho.$$

With N = 0, Eq. (4) becomes identical with the well-known equation of momentum conservation in conventional jets.

We now introduce a stream function defined in the following form:

$$\psi=A\frac{x}{\delta}f(\eta), \qquad \eta=B\frac{y}{\delta}, \tag{5}$$

where A and B are constants, $\delta=\delta(x)$ is a function of x to be determined. Eqs. (1), (2), and (5) yield the following relation:

$$f'^2\left(1-\frac{2x\delta'}{\delta}\right)-ff''\left(1-\frac{x\delta'}{\delta}\right)=\frac{\nu B}{A}f'''-\frac{N\delta^2}{AB}f', \tag{6}$$

where $\delta'=d\delta/dx$.

Under these conditions the function $\delta(x)$ can be found from Eq. (4),

$$\frac{x\delta'}{\delta}=\frac{2}{3}+\frac{N(x)2b}{3ABa}\delta^2, \tag{7}$$

whose general solution can be written as

$$\delta=\frac{x^{2/3}}{\left[D-\frac{4b}{3ABa}\int x^{1/3}N(x)\,dx\right]^{1/2}},$$

where

$$a=\int\limits_{-\infty}^{+\infty} f'^2 d\eta; \quad 2b=\int\limits_{-\infty}^{+\infty} f' d\eta = 2f(\infty).$$

Finally, assuming vB/A = 1 and using Eq. (7), we can rewrite Eq. (6) in the following form:

$$\left[f'''+\frac{1}{3}(ff''+f'^2)\right]=\frac{N(x)\delta^2(x)}{AB}\left[f'-\frac{4b}{3a}f'^2+\frac{2b}{3a}ff''\right]. \tag{8}$$

If N = 0, Eq. (8) becomes identical to the well-known equation of a two-dimensional jet in conventional hydrodynamics.

Since f is a function of η only, and the expression $N(x)\,\delta^2(x)/AB$ depends only upon the x coordinate, there are two possibilities:

1) The expressions in brackets on the left and right sides of Eq. (8) vanish simultaneously, i.e.,

$$f'''+\frac{1}{3}(ff''+f'^2)=0, \tag{8'}$$

$$f'-\frac{4b}{3a}f'^2+\frac{2b}{3a}ff''=0, \tag{8''}$$

with

$$f(0)=f''(0)=0, \quad f(\infty)=b. \tag{8'''}$$

Obviously, in the general case it is impossible to satisfy two different differential equations with the same function. However, in the case considered it is possible for the function $f = b\,\mathrm{th}\,b\eta/6$ to satisfy both Eqs. (8') and (8'') and also the boundary conditions (8'''), if one sets $a=2b^3/9$.

It can readily be seen that

$$a=\int\limits_{-\infty}^{+\infty} f'^2 d\eta = 2b^3/9.$$

It can further be assumed, without loss of generality, that $2b^3/9=1$, $b=\sqrt[3]{4.5}$.

If B(x) = const, i.e., the magnetic field is uniform, the constants A and B can be determined from the given initial value of the jet momentum (in the section x = 0):

$$\lim_{x\to 0}\rho\int\limits_{-\infty}^{+\infty} u^2 dy = I_0;$$

hence

$$\rho A^2 B \lim_{x\to 0}\frac{x^2}{\delta^3}=I_0.$$

We further require that

$$\lim_{x\to 0}\frac{x^2}{\delta^3}=1; \tag{9}$$

the corresponding values of A and B are then $A=\sqrt[3]{\frac{\nu I_0}{\rho}}, \quad B=\sqrt[3]{\frac{I_0}{\nu^2\rho}}.$

Hence the solution for the stream function can be written in the following form

$$\psi=1.655\sqrt[3]{\frac{\nu I_0}{\rho}}\frac{x}{\delta}\,\mathrm{th}\left(0.2755\sqrt[3]{\frac{I_0}{\nu^2\rho}}\frac{y}{\delta}\right),$$

where $\delta(x)$ is given by the equation

$$\frac{x\delta'}{\delta}=\frac{2}{3}+1.1\sigma B^2\sqrt[3]{\frac{\nu}{\rho I_0^2}}\,\delta^2 \qquad (10)$$

and is found to be equal to

$$\delta=\frac{Dx^{2/3}}{\sqrt{1-DN(4.5\nu\rho^2/I_0^2)^{1/3}x^{4/3}}}. \qquad (11)$$

The constant of integration D is found from the condition (9), which yields D = 1.

As can be seen from (11), for a given value of the magnetic field strength there exists a point on the axis of the jet in the neighborhood of which the jet disintegrates completely. The location of this point can be found from the relation

$$1-N\left(\frac{4.5\nu\rho^2}{I_0^2}\right)^{1/3}x^{4/3}=0.$$

Let us find now the total flow over the cross section of the jet

$$Q=\int_{-\infty}^{+\infty}u\,dy=2\left(4.5\frac{I_0^2x}{\nu\rho^2}\right)^{1/3}\sqrt{1-N\left(\frac{4.5\nu\rho^2}{I_0^2}\right)^{1/3}x^{4/3}}.$$

As can be seen, $Q(x)$ increases with x from zero to a maximum reached at a distance

$$x=(3N)^{-3/4}(I_0^2/4.5\nu\rho^2)^{1/4},$$

and then decreases to zero again. Hence, unlike the case of conventional hydrodynamic jets, in the case of magnetohydrodynamic jet flows in constant magnetic fields there exist two vortex zones symmetrically located with respect to the middle of the jet. The explanation of this phenomenon is as follows. The decelerating action of the magnetic field is still weak on the initial section of the jet, the jet entrains additional fluid masses from the environment, and the volume flow increases. At large distances from the source, the decelerating effect of the magnetic field becomes significant, ejection ceases, and a reverse process sets in, i.e., fluid is expelled from the jet into the adjacent region. The expelled fluid drifts to infinity and re-enters the jet zone at the source point.

2) Since $u > 0$, $AB > 0$ and $\frac{N}{AB}\delta^2=\text{const}=\alpha>0$. Under these conditions, Eq. (8) can be rewritten in the fol- form:

$$f'''+\left(\frac{1}{3}-\frac{2b}{3a}\alpha\right)ff''+\left(\frac{1}{3}+\frac{4b}{3a}\alpha\right)f'^2-\alpha f'=0.$$

The boundary conditions are given by (8‴).

Assuming that $P = f'$ is a function of f, we introduce a new function

$$\Phi=P+\frac{1}{6}f^2-\frac{1}{6}b^2,$$

assuming further that $a=2b^3/9$, we obtain an equation for $\Phi(f)$:

$$\left(\Phi-\frac{1}{6}f^2+\frac{1}{6}b^2\right)\Phi''+\left(\Phi'-\frac{1}{3}f-\frac{2b}{3a}\alpha\right)\Phi'+\frac{4b}{3a}\alpha\Phi=0$$

with the corresponding boundary conditions $\Phi(0)=\Phi'(0)=0$. This is identical with the Cauchy problem with homogeneous boundary conditions, whose obvious solution is $\Phi\equiv 0$, from which it follows that $f'=\frac{1}{6}b^2-\frac{1}{6}f^2$, and thus $f=b\,\mathrm{th}\frac{b\eta}{6}$. Hence the solution is the same for this case too.

It follows from Eq. (7) that in this case (i.e., with $N\delta^2$ = const)

$$\delta=x^{2/3+2b\alpha},\qquad N=\frac{N_0}{x^{4/3+4b\alpha}},\qquad N_0=\mathrm{const}.$$

(The case corresponding to this magnetic field configuration was considered by Jungclaus [1].)

For flows of this type the momentum in the initial section of the jet is infinite (condition (9) cannot be satisfied). Nevertheless, an invariant can also be found in this case, which permits the determination of the constants A and B. It can readily be seen that the quantity QI^n, where $n=({}^1/_3-2b\alpha)/6b\alpha$, is such an invariant.

The solution obtained indicates that it is possible to control the form of the jet by a suitable selection of the magnetic field configuration. For example, to obtain a jet whose shape is given by the parabola $\delta=kx^m$, it is necessary to apply a magnetic field of the form $B=B_0/x^m$. It follows then from Eq. (7) that, in accordance with [1], $m\geq 2/3$. In this case the quantity QI^n with $n=(m-1)/(2-3m)$ is invariant. In particular, in the case of a linear jet the volume flow Q is a constant along the entire length of the jet. If $m=2/3$, it follows from Eq. (7) that $B=0$ and I = const., i.e., this corresponds to a conventional jet with zero magnetic field.

2. Two-Dimensional Unbounded Turbulent Jet in a Transverse Magnetic Field

All the assumptions made in the laminar case apply here too. Furthermore, it will be assumed that the friction in the fluid can be described in terms of the modified Prandtl law:

$$\tau_T=k(x)\,u_m\frac{\partial u}{\partial y},$$

where k(x) is some function of x which reduces to $k(x)=Cx$ in the absence of a magnetic field. The quantity C is an experimental constant, u_m is the maximum velocity at the axis of the jet. Since the form of k(x) corresponding to the presence of an applied magnetic field is unknown, and there are no experimental data available, we shall use the similarity condition for the determination of the function.

Hence the statement of the problem reduces to the following set of equations:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=k(x)\,u_m\frac{\partial^2 u}{\partial y^2}-Nu,\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{12}$$

with boundary conditions identical with those specified for the laminar case. We shall seek a stream function of the following form

$$\psi=A\frac{x^{3/2}}{\delta}f\left(B\frac{y}{\delta}\right),$$

where $\delta=\delta(x)$ is the half-width of the jet, a quantity to be determined together with f. Under these conditions, Eq. (12) and condition (4), which is also valid for turbulent flows, can be rewritten in the following form

$$f'^2\left[\frac{3}{2}-2\frac{x\delta'}{\delta}\right]+f''f\left[-\frac{3}{2}+\frac{x\delta'}{\delta}\right]=\frac{k(x)B^2x}{\delta^2}f'''-\frac{N}{AB}\frac{\delta^2}{x^{1/2}}f' \tag{13}$$

and

$$\frac{x\delta'}{\delta}=1+\frac{Na}{3ABb}\frac{\delta^2}{x^{1/2}}. \tag{14}$$

If one requires

$$\frac{k(x)B^2x}{\delta^2}=1 \tag{15}$$

and uses the relation (14), Eq. (13) becomes

$$f''' + \frac{1}{2}(f'^2 + ff'') = \frac{N\delta^2}{ABx^{1/2}}\left(-f' + \frac{2a}{3b}f'^2 - \frac{a}{3b}ff''\right).$$

Hence we arrive at an equation similar to the one discussed above, whose solution can be written as

$$f = \sqrt{4C \,\mathrm{th}\, \sqrt{C/4}\,\eta}.$$

The constants A, B, and C are determined as follows: if the momentum is given in the initial section of the jet $\rho\int_{-\infty}^{+\infty} u^2 dy = I_0$, and $A^2B^2 = I_0/\rho$. If one assumes that $\int_{-\infty}^{+\infty} f'^2\, d\eta = 1$, then $C = \sqrt[3]{\frac{9}{64}}$. The remaining constant B must be found experimentally.

As in the laminar case, the half-width of the jet is found from Eq. (14), which can be rewritten in a slightly different form:

$$\frac{d}{dx}\frac{x^3}{\delta^3} = -N_0\frac{x^{3/2}}{\delta}, \tag{16}$$

where $N_0 = N\sqrt{\rho/I_0}\sqrt[3]{24}$.

The solution of Eq. (16) corresponding to an arbitrary function can be found by introducing

$$\frac{d}{dx}T^3 = -N_0(x)x^{1/2}T, \quad \text{hence} \quad \delta^2 = \frac{x^2}{D - \frac{4}{9}\int N_0(x)x^{1/2}dx}.$$

In particular, if $N_0(x) = \mathrm{const} = N_0$

$$\delta^2 = \frac{x^2}{D - \frac{4}{9}N_0x^{3/2}}, \tag{17}$$

and D is found from the condition that at N = 0 δ = x; thus D = 1.

The function k(x) can now be found from Eq. (15)

$$k(x) = \frac{1}{B^2}\frac{\delta^2}{x} = \frac{1}{B^2}\frac{x}{1 - \frac{4}{9}N_0x^{3/2}}.$$

Clearly, k(x) has the same shortcomings as the mixing length in the absence of a magnetic field: the initial condition for the mixing length is that it is small compared with the flow region, while the solution shows that k(x) is proportional to the width of the jet. Since in a magnetic field the expansion of the jet is defined by Eq. (17), the function k(x) has the same characteristics as the width of the mixing zone.

As in the case of a laminar flow, it is possible to control the shape of a turbulent jet by changing the configuration of the applied magnetic field. For example, to obtain a jet with $\delta = \alpha x^m$, the application of a field of the form $B = \frac{B_0}{x^{m-1/4}}$ is necessary. Under these conditions, the constants of integrations are found from invariants of the form

$$IQ^n = \mathrm{const}\left(n = \frac{3(m-1)}{3/2 - m}\right) \quad \text{or} \quad I^nQ = \mathrm{const}\left(n = \frac{m - 3/2}{3(1-m)}\right).$$

The case m = 1 corresponds to a hydrodynamic jet (N = 0), the case m = 3/2 to constant volume flow along the jet. The condition (14) imposes a restriction upon the magnitude of m:

$$m \geqslant 1.$$

3. Laminar Radial Slit Jets

Similar considerations can also be applied to radial slit jets, in which case three velocity components are generally present (twisted jets). The system of equations that defines these jets can be written in cylindrical coordinates as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{w^2}{x}=\nu\frac{\partial^2 u}{\partial y^2}-Nu;\quad u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{uw}{x}=\nu\frac{\partial^2 w}{\partial y^2}-Nw;\quad \frac{\partial xu}{\partial x}+\frac{\partial xw}{\partial y}=0, \tag{18}$$

where the corresponding boundary conditions are $\partial u/\partial y = v = \partial w/\partial y = 0$ at $y = 0$; $u = w = 0$ at $y \to \pm\infty$. This system yields a set of integral relations which expresses the radial component of the momentum and of the moment of momentum in the following form:

$$\frac{d}{dx}\int_{-\infty}^{+\infty} xu^2 dy=\int_{-\infty}^{+\infty} w^2 dy - N\int_{-\infty}^{+\infty} xu\,dy; \tag{19}$$

$$\frac{d}{dx}\int_{-\infty}^{+\infty} x^2 uw\,dy=-N\int_{-\infty}^{+\infty} x^2 w\,dy. \tag{20}$$

Following the usual procedure, we introduce a stream function defined as

$$\psi=\sqrt{\nu}\,x^2\left(\frac{f_1}{\delta}+\frac{f_2}{\delta^2}+\frac{f_3}{\delta^3}+\dots\right)$$

and a swirl velocity

$$w=x\left(\frac{\varphi_1}{\delta^2}+\frac{\varphi_2}{\delta^3}+\dots\right),$$

where $f_i=f_i(\eta)$, $\varphi_i=\varphi_i(\eta)$, $\eta=\dfrac{y}{\sqrt{\nu}\,\delta}$.

Hence we get a system of equations for the successive approximations

$$\begin{gathered}
f'^2_1\left(1-\frac{2x\delta'}{\delta}\right)+f''_1 f_1\left(\frac{x\delta'}{\delta}-2\right)-\varphi_1^2=f'''_1-N\delta^2 f'_1,\\
f'_1 f'_2\left(2-\frac{5x\delta'}{\delta}\right)+f''_1 f_2\left(\frac{2x\delta'}{\delta}-2\right)+f''_2 f_1\left(\frac{x\delta'}{\delta}-2\right)-2\varphi_1\varphi_2=\\
=f'''_2-N\delta^2 f'_2,\\
\dots\dots\dots\dots\dots\dots\dots\dots\\
f'_1\varphi_1\left(2-\frac{2x\delta'}{\delta}\right)+\varphi'_1 f_1\left(\frac{x\delta'}{\delta}-2\right)=\varphi''_1-N\delta^2\varphi_1,\\
f'_1\varphi_2\left(2-\frac{3x\delta'}{\delta}\right)-\varphi_1 f'_2\left(2-\frac{2x\delta'}{\delta}\right)=\varphi'_1 f_2\left(\frac{2x\delta'}{\delta}-2\right)+f_1\varphi'_2\left(\frac{x\delta'}{\delta}-2\right)=\\
=\varphi''_2-N\delta^2\varphi_2.
\end{gathered} \tag{20'}$$

The corresponding boundary conditions are $f_1=f_2=f''_1=f''_2=0$, $\varphi'_1=\varphi'_2=0$ at $\eta=0$; $f'_1=f'_2=0$, $\varphi_1=\varphi_2=0$ at $\eta = \infty$.

Under these conditions Eqs. (19), (20) can be rewritten in the following form:

$$\begin{gathered}
3\left(1-\frac{x\delta'}{\delta}\right)\int_{-\infty}^{+\infty} f'^2_1 d\eta=\int_{-\infty}^{+\infty}\varphi_1^2 d\eta-N\delta^2\int_{-\infty}^{+\infty} f'_1 d\eta,\\
\left(4-\frac{3x\delta'}{\delta}\right)\int_{-\infty}^{+\infty} f'_1\varphi_1 d\eta=-N\delta^2\int_{-\infty}^{+\infty}\varphi_1 d\eta,\\
\left(3-\frac{2x\delta'}{\delta}\right)\int_{-\infty}^{+\infty} f'_1 f'_2 d\eta=\int_{-\infty}^{+\infty}\varphi_1\varphi_2 d\eta-\frac{N\delta^2}{2}\int_{-\infty}^{+\infty} f'_2 d\eta,
\end{gathered} \tag{21}$$

$$4\left(1-\frac{x\delta'}{\delta}\right)\int_{-\infty}^{+\infty}(f'_2\varphi_1+f'_1\varphi_2)\,d\eta=-N\delta^2\int_{-\infty}^{+\infty}\varphi_2 d\eta. \qquad (21)\ \text{(cont'd)}$$

We introduce the following notation:

$$\int_{-\infty}^{+\infty}f'^2 d\eta=a,\quad \int_{-\infty}^{+\infty}\varphi_1^2 d\eta=b,\quad \int_{-\infty}^{+\infty}f'_1 d\eta=c,\quad \int_{-\infty}^{+\infty}f'_1\varphi_1 d\eta=d,$$

$$\int_{-\infty}^{+\infty}\varphi_1 d\eta=k,\quad \int_{-\infty}^{+\infty}f'_1 f'_2 d\eta=m,\quad \int_{-\infty}^{+\infty}\varphi_1\varphi_2 d\eta=n,$$

$$\int_{-\infty}^{+\infty}f'_2 d\eta=p,\quad \int_{-\infty}^{+\infty}(f'_2\varphi_1+f'_1\varphi_2)\,d\eta=q,\quad \int_{-\infty}^{+\infty}\varphi_2 d\eta=h.$$

It follows from the first of Eqs. (21) that

$$\frac{x\delta'}{\delta}=\left(1-\frac{b}{3a}\right)+\frac{Nc}{3a}\delta^2. \qquad (22)$$

Substituting (22) into the second expression in (21), we obtain d = 0 and k = 0 (provided $a \neq 0$).

As is known [2], $\varphi_1 \equiv 0$ in the absence of magnetic fields. We shall prove that this is the case also in the presence of a magnetic field. For this purpose we substitute Eq. (22) into the third equation of (20'):

$$\frac{2b}{3a}f'_1\varphi_1-\left[1+\frac{b}{3a}\right]\varphi'_1 f_1-\varphi''_1=-N\delta^2\left[\varphi_1-\frac{2c}{3a}f'_1\varphi_1+\frac{c}{3a}\varphi'_1 f_1\right].$$

Considerations analogous to those advanced in the case of two-dimensional jets lead to the differential equation

$$\varphi'_1 f_1-2f'_1\varphi_1+(3a/c)\,\varphi_1=0,$$

which, after integration, yields the following solution: $\varphi_1=Af_1^2\exp\left(-\frac{3a}{c}\int\frac{d\eta}{f_1}\right)$, i.e., φ_1 does not change sign. Recalling that $k=\int_{-\infty}^{\infty}\varphi_1 d\eta=0$, we obtain $\varphi_1 \equiv 0$. Hence b = n = m = p = 0, 4cq = 3ah, and $x\delta'/\delta = 1 + (Nc/3a)\delta^2$. The general solution of the last equation is represented by

$$\delta^2=\frac{x^2}{1-\frac{2c}{3a}\int N(x)\,x\,dx}.$$

Substituting (22) into the first and the third equations of (20'), we obtain equations analogous to (8). The solutions of these equations yield the following expressions for the first-order velocity components:

$$u=\frac{\alpha^2\left(1-\frac{2c}{3a}\int N(x)\,x\,dx\right)}{2x\,\mathrm{ch}^2\left(\frac{\alpha\eta}{2}\right)},$$

$$v=\frac{\sqrt{\nu}\,\alpha\left(1-\frac{2c}{3a}\int N(x)\,x\,dx\right)^{1/2}}{2x}\left[\frac{\eta\alpha-\mathrm{sh}\,(\alpha\eta)}{\mathrm{ch}^2(\alpha\eta/2)}+\frac{N(x)\,c}{3a}\,\frac{x^2}{1-\frac{2c}{3a}\int N(x)\,x\,dx}\,\frac{\eta\alpha+\mathrm{sh}\,(\alpha\eta)}{\mathrm{ch}^2(\alpha\eta/2)}\right],$$

$$w=\frac{1}{2}\sqrt[3]{\frac{9}{16\pi^2}\,\frac{L_0}{\sqrt[3]{\mu\rho I_0}}}\,\frac{1}{x^2}\left[1-\frac{2c}{3a}\int N(x)\,x\,dx\right]^{3/2}\frac{1}{\mathrm{ch}^2(\alpha\eta/2)},$$

where $\alpha = \sqrt[3]{\frac{3I_0}{4\pi\rho\sqrt{\nu}}}$; I_0 and L_0 correspond to the radial momentum at $x \to \infty$, $N \to 0$ and to the moment of momentum at $N \to 0$, respectively.

With $w = 0$ ($L_0 = 0$), we obtain the solution for a radial slit jet with zero swirl.

In conclusion, we note the following feature of the solutions considered. As has been shown, in the case of jets in magnetic fields there exists a section in the neighborhood of which the longitudinal velocity reduces to zero and the transverse velocity component increases indefinitely. The corresponding boundary layer thickness also increases without bound. Since the boundary layer approximation is based on the assumption of small transverse velocities and finite boundary layer thicknesses, we must conclude that the boundary layer equations are inapplicable in the neighborhood of this section.

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