# SELF-EXCITATION OF A MAGNETIC FIELD BY A PAIR OF ANNULAR EDDIES

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The stationary excitation of a magnetic field in a system of two eddies having an axisymmetric velocity distribution is investigated in the case when the dimension of the eddy is commensurate with the dimensions of the overall system. The relative dimensions of the model which are optimal for self-excitation are found. It is established that fields both even and odd with respect to reflection in the symmetry plane are excited for values of Rm that are identical in modulus.

In [1] it was shown that an axisymmetric pair of stationary annular eddies in an unbounded homogeneous electrically conducting incompressible fluid can excite a magnetic field. The dependence of the field on the azimuth $\varphi$ turns out to be harmonic:

$$B \sim e^{im\varphi}, \quad m=1, 2, 3, \ldots \tag{1}$$

The critical magnetic Reynolds number

$$\mathrm{Rm} \equiv \mu_0 \sigma \int_0^R (r/R)^2 v(r)\, dr = \pm \frac{2Z_0^2}{mR^2} \left[ \oint \frac{\sin\varphi \sin m\varphi d\varphi}{[1+(2aZ_0^{-1} \sin {}^1\!/_2\varphi)^2]^{3/2}} \right]^{-1} \tag{2}$$

for the given harmonic m depends on the system dimensions illustrated in Fig. 1.

Condition (2) is applicable for an arbitrary dependence of the velocity v(r) in the eddy along the minor radius of the eddy $0 \le r \le R$, but only on the assumption that

$$R \ll \min(Z_0, a), \tag{3}$$

(i.e., (2) is an asymptotic formula that is applicable only for thin eddies when the self-excitation develops for large Rm).

For increasing R the value of Rm decreases and reaches a minimum outside the domain (3), so that one cannot find the dimensions of the most easily excited eddies from (2).

The finding of the relative dimensions of the system which excites the magnetic field most easily and corresponds to the smallest Rm is a fundamental problem which precedes an experimental investigation of a homogeneous dynamo. For this purpose the constraint (3) is not used in this present paper, but a special form of the velocity distribution is investigated.

It is assumed that the motion is concentrated in a thin layer on the surface of two tori (layer thickness $\delta \ll R$), while the remaining fluid is stationary. Just as in [1], it is assumed that motion takes place only in planes passing through the symmetry axis, is independent of the azimuth $\varphi$, and is oppositely directed in the two eddies.

The critical Rm number is defined as the eigenvalue of a certain integral equation. In order to derive the latter the expression for the current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{[(\mathbf{r}'-\mathbf{r}) \times \mathbf{j}(\mathbf{r}')]}{|\mathbf{r}'-\mathbf{r}|^3}\, d^3r'$$

is substituted into the Biot–Savart law

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$$j=\sigma(-\mathrm{grad}\,\Phi+\mathbf{v}\times\mathbf{B}).$$

For integration by parts the term containing the electrostatic potential $\Phi$ is eliminated. As a result,

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0\sigma}{4\pi}\int\frac{(\mathbf{r}'-\mathbf{r})\times[\mathbf{v}(\mathbf{r}')\times\mathbf{B}(\mathbf{r}')]}{|\mathbf{r}'-\mathbf{r}|^3}d^3r' \tag{4}$$

is obtained. In (4) the integration domain is only the space having $\mathbf{v}\neq 0$ (i.e., the surface layer of the two tori). Only two equations are essential in the system (4), since the magnetic field component parallel to the velocity vector is not contained in the product $\mathbf{v}\times\mathbf{B}$. By virtue of the axial symmetry of the system the dependence of the solutions $\mathbf{B}(\mathbf{r})$ on $\varphi$ may only be harmonic (1), while by virtue of symmetry of the system with respect to reflection,the solutions in the horizontal plane between eddies may only be even or odd relative to this reflection (see Appendix).

Let us use the cylindrical coordinate system ($\rho$, $\varphi$, z) with the z axis in the direction of the symmetry axis of the system and a plane z = 0 coinciding with the reflected-symmetry plane of the system: $\mathbf{B}=(B_\rho, B_\varphi, B_z)$, $\mathbf{v}=(v_\rho, 0, v_z)$. Let us stipulate the velocity in such a way that $v=v_0\cdot a/\rho$, where $v_0$ is constant. The solution of the system (4) shall be sought in the form

$$B_r(\mathbf{r})\equiv(v_\rho B_z-v_zB_\rho)/v\equiv\tilde{v}_\rho B_z-\tilde{v}_zB_\rho=b_r(\varrho,z)\cdot e^{im\varphi},$$
$$B_\varphi(\mathbf{r})=b_\varphi(\varrho,z)\cdot e^{im\varphi}. \tag{5}$$

Substituting (5) into (4) and integrating with respect to $\varphi$, we obtain the system of equations

$$b_r(\mathbf{x})=\frac{\mathrm{Rm}}{4\pi R\delta}\int[K_{rr}(\mathbf{x},\mathbf{x}')\cdot b_r(\mathbf{x}')+K_{r\varphi}(\mathbf{x},\mathbf{x}')\cdot b_\varphi(\mathbf{x}')]\,d^2x', \tag{6}$$
$$b_\varphi(\mathbf{x})=\frac{\mathrm{Rm}}{4\pi R\delta}\int[K_{\varphi r}(\mathbf{x},\mathbf{x}')\cdot b_r(\mathbf{x}')+K_{\varphi\varphi}(\mathbf{x},\mathbf{x}')\cdot b_\varphi(\mathbf{x}')]\,d^2x',$$

for $b_r$ and $b_\varphi$, where

$$\mathbf{x}=(\varrho,z),\quad d^2x'=d\varrho'dz',$$
$$K_{rr}(\mathbf{x},\mathbf{x}')\equiv K_{rr}(\varrho,z,\varrho',z')=[\varrho\tilde{v}_\rho-(z'-z)\tilde{v}_z]D(\mathbf{x},\mathbf{x}')+\varrho'\tilde{v}_\rho C(\mathbf{x},\mathbf{x}'),$$
$$K_{r\varphi}(\mathbf{x},\mathbf{x}')=[\varrho\tilde{v}'_z,\ \tilde{v}_\rho-\varrho'\tilde{v}'_\rho,\ \tilde{v}_z-(z'-z)\tilde{v}'_z\tilde{v}_z]\,S(\mathbf{x},\mathbf{x}'),$$
$$K_{\varphi r}(\mathbf{x},\mathbf{x}')=(z'-z)\,S(\mathbf{x},\mathbf{x}'),$$
$$K_{\varphi\varphi}(\mathbf{x},\mathbf{x}')=\varrho\tilde{v}'_\rho C(\mathbf{x},\mathbf{x}')-[\varrho'\tilde{v}'_\rho+(z'-z)\tilde{v}]\,D(\mathbf{x},\mathbf{x}'),$$
$$C(\mathbf{x},\mathbf{x}')=R(a-R)\oint\frac{\cos m\varphi\,d\varphi}{|\mathbf{r}'-\mathbf{r}|^3},$$
$$D(\mathbf{x},\mathbf{x}')=R(a-R)\oint\frac{\cos\varphi\cdot\cos m\varphi\,d\varphi}{|\mathbf{r}'-\mathbf{r}|^3},$$
$$S(\mathbf{x},\mathbf{x}')=R(a-R)\oint\frac{\sin\varphi\cdot\sin m\varphi\,d\varphi}{|\mathbf{r}'-\mathbf{r}|^3},$$
$$\mathrm{Rm}=\mu_0\sigma\delta v_{\max}=\mu_0\sigma\delta v_0/(1-R/a).$$

Rm is determined from the maximum velocity. In a toroidal eddy the velocity of the incompressible fluid has a maximum at the point closest to the symmetry axis.

In (6) integration is performed over two rings (radius R and thickness $\delta$) which are formed by the intersection of the $\varphi$ = const plane with the moving layers. Using the parity of the solution, we assign the integration domain to one eddy. By virtue of $\delta \ll R$ (6) can be integrated trivially over the thickness of the ring and reduced to a system of one-dimensional integral equations:

$$b_r(\chi)=\frac{\mathrm{Rm}}{4\pi}\oint[Q_{rr}(\chi,\chi')b_r(\chi')+Q_{r\varphi}(\chi,\chi')b_\varphi(\chi')]\,d\chi', \tag{7}$$
$$b_\varphi(\chi)=\frac{\mathrm{Rm}}{4\pi}\oint[Q_{\varphi r}(\chi,\chi')b_r(\chi')+Q_{\varphi\varphi}(\chi',\chi)b_\varphi(\chi')]\,d\chi'.$$

Here

$$Q_{\alpha\beta}(\chi,\chi')=K_{\alpha\beta}(\varrho,z,\varrho',z')+s_{\alpha\beta}K_{\alpha\beta}(\varrho,z,\varrho',(Z_0-z')),$$

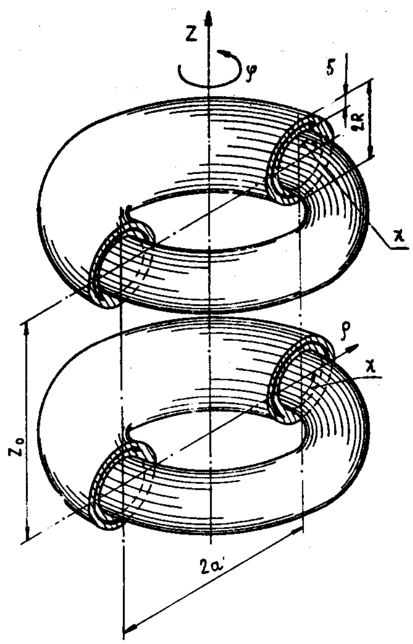


Fig. 1. Diagram of the model.

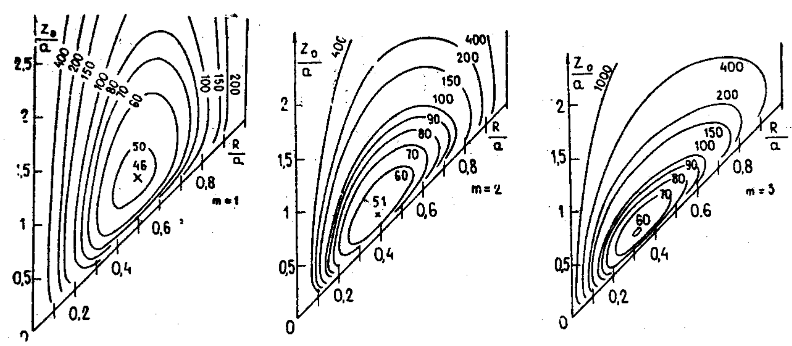


Fig. 2. Level lines of the critical magnetic Reynolds number Rm.

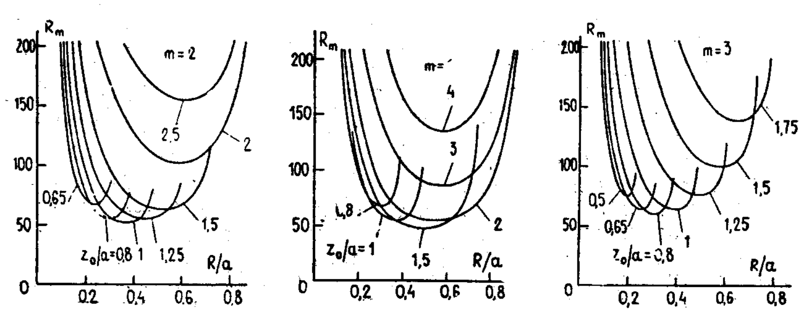


Fig. 3. Dependence of the Rm number on the minor radius of the eddy.

where $\alpha$, $\beta \in (r, \varphi)$, $\chi$ is the angular coordinate in the polar coordinate system with its center at the center of the integration ring (see Fig. 1).

The sign multiplier s takes the values

$$s_{rr}=s_{\varphi r}=-1, \quad s_{r\varphi}=s_{\varphi\varphi}=1$$

TABLE 1

| $m$ | $Z_0/a$ | $R/a$ | Rm |
|---|---|---|---|
| 1 | 1,47 | 0,49 | 46 |
| 2 | 0,98 | 0,37 | 51 |
| 3 | 0,79 | 0,31 | 59 |

for the even solution and the values

$$s_{rr}=s_{\varphi r}=1,\quad s_{r\varphi}=s_{\varphi\varphi}=-1$$

for the odd solution.

The problem of eigenvalues for the system (7) was solved empirically on an electronic computer. The integrals in (7) were replaced by finite sums in quadrature formulas with equally spaced points. For the chosen relative dimensions of the model it was sufficient to use formulas with several tens of nodal points. The greatest real eigenvalue $Rm^{-1}$ and the corresponding eigensolution for the derived algebraic system were found by the iteration interpolation method (see [2]).

The system (7) is a system of singular integral equations with floating singularities of the type $(\chi'-\chi)^{-1}$ and $\ln(\chi'-\chi)$. The singularities were handled with allowance (in the form of an additive term) for the behavior of the kernels $Q_{rr}$ and $Q_{\varphi\varphi}$ for $\chi'\to\chi$ (see [3]). The integrals C, D, and S were expressed in terms of power-law series. For economization (see [3]) of the latter, approximating polynomials were obtained that were used in the computation.

The results of the calculation for the first three harmonics are displayed in Figs. 2 and 3. Our model corresponds to the domain $0 < R < \min(Z_0/2, a)$ which is what is shown in the figures. In all three cases for the critical Rm number explicitly defined minima are obtained whose coordinates are given in Table 1.

It follows from the results obtained that in order to achieve the excitation of the higher harmonics it is necessary to bring the eddies closer together and to reduce their minor radius. Under these conditions the required Rm number increases. Moreover, at the minimum point for the field having $m=3$ the field having $m=2$ is excited for a still lower $Rm=56$. Therefore, in order to excite only the field with $m=3$ the required smallest number $Rm=70$ is obtained at the point $Z_0/a=0.57$, $R/a=0.22$.

In the domain (3) the numerical calculations coincide with Eq. (2). The quantities $Rm(R/a)^2$ depicted in Fig. 4 were likewise calculated in [1] in the asymptotic limit $R/a\to 0$. From Fig. 4 it is evident that the difference between Rm and the asymptotic value of (2) increases with an increase in the radius R of the eddy. For $R/a=0.1$ this difference is already about 20%. It likewise increases with an increasing number m.

In conclusion, note that in order to attain the critical values of Rm in the laboratory model one requires a high hydraulic power. The hydraulic power required to maintain motion in our model is estimated according to the equation

$$W=2\pi\zeta(a-R)\frac{\gamma}{\delta^2}\left(\frac{Rm}{\mu_0\sigma}\right)^3, \tag{8}$$

where $\gamma$ is the specific density of the medium; $\zeta=P/{}^1\!/_2\gamma v^2$ is the coefficient of hydraulic resistance; P is the pressure differential.

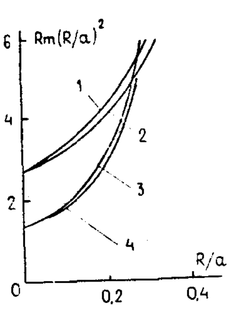


Fig. 4

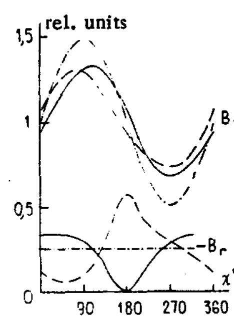


Fig. 5

Fig. 4. Deviation of Rm from the asymptotic value. For the chosen values of $Z_0/a$ curves having different m intersect in the asymptotic limit $R/a\to 0$.

Fig. 5. Field distribution for the minimal $Rm=46$ and $m=1$. The solid curve is the even solution; the dashed curve is the odd solution; the dash–dot curve is the field for complete freezing-in.

Substituting the following numerical values (in the Engineering System system of units) in (8): $\zeta = 0.8$ (estimated from [4]); Rm = 46; $R/a = 0.49$ (the point corresponding to the minimum Rm); $\sigma = 1.03 \cdot 10^7$; $\gamma = 928$ (Na at 100°C); $\mu_0 = 4\pi \cdot 10^{-7}$; $a = 1$; $\delta = 0.1$, we obtain W = 12 MW.

## APPENDIX

Excitation of the even and odd fields.

The kernels of the integral equations (6) have the following symmetry:

$$K_{rr}(z, z') = K_{rr}(-z, -z'), \quad K_{r\varphi}(z, z') = -K_{r\varphi}(-z, -z'),$$
$$K_{\varphi r}(z, z') = -K_{\varphi r}(-z, -z'), \quad K_{\varphi\varphi}(z, z') = K_{\varphi\varphi}(-z, -z'); \tag{9}$$

$$K_{rr}(\mathbf{x}, \mathbf{x}') = -K_{\varphi\varphi}(\mathbf{x}', \mathbf{x}), \quad K_{r\varphi}(\mathbf{x}, \mathbf{x}') = -K_{r\varphi}(\mathbf{x}', \mathbf{x}),$$
$$K_{\varphi r}(\mathbf{x}, \mathbf{x}') = -K_{\varphi r}(\mathbf{x}', \mathbf{x}). \tag{10}$$

Assume that for a certain Rm a nontrivial solution of the system (6) ($b_r$, $b_\varphi$) has been solved. Then, reversing the direction of the z axis and using Eq. (9), we find that for the same Rm the functions ($-b_r(-z)$, $b_\varphi(-z)$) are also a solution. In such a case the absence of degeneracy leads to the requirement

$$(-b_r(-z),\ b_\varphi(-z)) = (sb_r(z),\ sb_\varphi(z)),$$

where s is a certain constant:

$$(b_r(z),\ b_\varphi(z)) = (-sb_r(-z),\ sb_\varphi(-z)) = (s^2 b_r(z),\ s^2 b_\varphi(z)). \tag{11}$$

Two cases are possible:

1) s = +1. The excited field is even with respect to reflection in the horizontal plane: after reflection the field **B** coincides with itself (**B**→**B**). The solution of the system (6) has symmetry:

$$b_r^+(z) = -b_r^+(-z); \quad b_\varphi^+(z) = b_\varphi^+(-z). \tag{12}$$

2) s = −1. The excited field is odd with respect to reflection (**B**→−**B**):

$$B_r^-(z) = b_r^-(-z); \quad b_\rho^-(z) = -b_\varphi^-(-z). \tag{13}$$

This is a well-known result: the symmetry properties of the linear operator lead to classification of the eigenfunctions according to these same symmetry properties.

Assume that the even ($b_r^+$, $b_\varphi^+$) and odd ($b_r^-$, $b_\varphi^-$) fields are excited for the magnetic Reynolds numbers $\mathrm{Rm}^+$ and $\mathrm{Rm}^-$, respectively. Using Eqs. (6), the properties of the kernels of (10), and the parity of the solutions (12) and (13), we obtain the relationship

$$\left(\frac{1}{\mathrm{Rm}^+} + \frac{1}{\mathrm{Rm}^-}\right) \int (b_r^+ b_\varphi^- + b_\varphi^+ b_r^-)\, d^2x = 0. \tag{14}$$

If the integral in (14) is nonvanishing, then $\mathrm{Rm}^+ = -\mathrm{Rm}^-$ (even and odd fields are excited at equal velocities but for opposite directions of motion). A sufficient condition for a nonvanishing integral in (14) is constancy of the sign of the function ($b_r^+$, $b_\varphi^+$), ($b_r^-$, $b_\varphi^-$) in the domain $\mathbf{v} \neq 0$ of one eddy, and an identical sign of the two terms. This condition is satisfied both in the asymptotic case [1] and in all empirically investigated cases in the present work (Fig. 5). The sign-constancy of the function ($b_r$, $b_\varphi$) is associated with the comparatively large values of Rm which cause the approximate freezing-in [5] of the field in the moving fluid.

For the directions of rotation shown in Fig. 1 an even field is excited, while for the opposite directions an odd field is excited.

## LITERATURE CITED


1. A. Gailitis, Magnitn. Gidrodinam., No. 1, 19 (1970).
2. J. H. Wilkinson, Algebraic Eigenvalue Problem, Oxford University Press (1965).
3. R. W. Hamming, Numerical Methods for Scientists and Engineers, McGraw-Hill (1962).
4. I. E. Idel'chik, Handbook on Hydraulic Resistances [in Russian], Moscow–Leningrad (1960).
5. L. D. Landau and E. M. Lifshitz, Electrodynamics of Continuous Media, Addison-Wesley (1960).