GENERATION OF A MAGNETIC FIELD IN A TURBULENT CONDUCTING FLUID

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It is shown that the rise time of a field satisfies a certain inequality having a clear physical meaning. A theory of magnetic instability similar to the Landau theory is constructed which describes hydrodynamic instability. A general equation for the generation of a regular magnetic field is derived.

1. The problem of generation of a magnetic field by a turbulently moving conducting fluid is usually treated in one of two arrangements: a) a dynamo of small-scale (random) fields, and b) a dynamo of large-scale and regular fields.

In its entirety this problem was first treated in [1] where a qualitative picture of the growth of the magnetic field was given. The growth of the field, as we know, is caused by an elongation of the magnetic lines of force; the authors of [1] assume that this elongation is entirely associated with the small-scale turbulence structure, so that

$$H = |\mathbf{H}| \sim e^{-\alpha t/T}, \tag{1.1}$$

where T is the period of the small-scale motion; α is a constant of the order of unity. The increasing magnetic field suppresses the small-scale pulsations whose kinetic energy is small in comparison with the field energy. The pulsations having large scales are not subject to the effect of the field, and due to these pulsations a further stretching of the lines of force takes place. Since the scale of the smallest pulsations increases, the quantity T in (1.1) also increases. If one assumes that uniform distribution is established (i.e., the magnetic energy coincides with kinetic energy in order of magnitude), then one may obtain certain inequalities which the settling period τ satisfies.

Let us use $T_{\lambda 0}$ and T_l to denote the periods of the largest and smallest turbulence scales, respectively. It is clear that the condition

$$t_{\lambda 0} < \tau < t_l \tag{1.2}$$

must hold, where $t_{\lambda 0}$ is the rise time of the field for $T = T_{\lambda 0}$, t_l is the rise time for $T = T_l$. We shall assume that the magnetic energy increases from an initial value $H_{\lambda 0}^2 \sim \rho v_{\lambda 0}^2$ to a final value $H_l^2 \sim \rho v_l^2$. The frequencies of the largest scales are $\omega_l \sim v_l/l$, the frequencies of the smallest scales are $\omega_{\lambda 0} \sim R^{3/4} v_l/l$ [2] (v_l is the characteristic velocity; $R = v_l l/\nu$). Since $T = \omega^{-1}$, we obtain

$$t_{\lambda 0} \simeq \frac{l}{v_l} R^{-3/4} \ln \frac{v_l}{v_{\lambda 0}}, \quad t_l \simeq \frac{l}{v_l} \ln \frac{v_l}{v_{\lambda 0}} \tag{1.3), (1.4}$$

from (1.1). But $v_l/v_{\lambda 0} \sim R^{1/4}$ [2], and (1.2) yields

$$\frac{1}{4} \frac{l}{v_l} R^{3/4} \ln R < \tau < \frac{1}{4} \frac{l}{v_l} \ln R. \tag{1.5}$$

The inequalities (1.5) show that the rise time of the magnetic field increases with a growth in the dimensions of the turbulence region and decreases with an increase in velocity. Physically this is associated with the following factors. The number of degrees of freedom n of the turbulent motion per unit volume of the fluid has the order [2]

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$$n\sim\frac{1}{l^3}R^{9/4}. \tag{1.6}$$

The magnitude of the energy "interval" between the extreme sectors of the spectrum is $\rho v_l^2-\rho v_{\lambda_0}^2\approx\rho v_l^2$, since $v_l\gg v_{\lambda_0}$. For an increase in l the length of the "interval" does not change, and the number of degrees of freedom n decreases as $l^{-3/4}$. The growth of the magnetic field is a process of interaction between different degrees of freedom which is conventional for turbulence. But if the distances between them increase, then the interaction weakens, which must lead to a slowing of such a process. Conversely, when the velocity increases, the distances between the degrees of freedom decrease, since n increases as $v^{9/4}$ while the length of the energy interval increases as v^2, which leads to a more rapid growth of the field.

Thus, the obtained inequalities are very clear in character and are evidently in qualitative agreement with the actual situation. They could be improved by introducing estimates which include a dimensionless parameter characterizing the electromagnetic properties of the medium – for example, Rm. However, the best results, which constitute (as is obvious from dimensionality concepts) an equation of the form

$$\tau=\frac{l}{v}f(R,\,R_m,\,Ha), \tag{1.7}$$

would have to be obtained for solution of the complete system of magnetohydrodynamic equations.

2. The problem of the existence of a turbulent dynamo of random fields currently has no unique answer. It was considered for the first time in the well-known paper by Batchelor [4] which, however, encountered a multitude of objections.

The process involving the development of the magnetic field in a moving fluid is evidently a result of the instability of motion relative to magnetic perturbations. As an example, let us begin by considering the stationary streaming of a conducting fluid past a body having finite dimensions. In this case magnetic instability which is formally analogous to absolute hydrodynamic instability [2, 3] is in general possible. If there is no initial stationary field, then one may get by with the induction equation

$$\frac{\partial\mathbf{H}}{\partial t}=\mathrm{curl}[\mathbf{vH}]+\nu_m\Delta\mathbf{H}, \tag{2.1}$$

since the equation of motion includes terms that are quadratic in **H**. The role of the dimensionless parameter is played by $R_m=vl/\nu_m$ (instead of R in the case of hydrodynamic instability).

H should be sought in the form

$$\mathbf{H}=\mathbf{h}(x,\,y,\,z)e^{-i\omega t}, \tag{2.2}$$

where **h** and ω are complex. If one can find ω such that $\beta=\Im m\,\omega>0$, such motion is unstable, and a magnetic field having the frequency $\alpha=\Re e\,\omega$ develops. For the mean-square modulus of the amplitude we obtain the following equation by analogy with the Landau theory of hydrodynamic instability [2, 3]:

$$\frac{d\overline{|A|^2}}{dt}=\beta|A|^2-\gamma|A|^4, \tag{2.3}$$

i.e., the limit of $|A|^2$ is

$$|A|_{max}^2=2\beta/\gamma. \tag{2.4}$$

Since in this case R_m is the dimensionless parameter, it follows that $\beta=\beta(R_m)$. From the very definition of the critical parameter $\beta(R_{m\,cr})=0$. Therefore, for low values of the difference $R_m-R_{m\,cr}$ we have the following result within terms of the first order:

$$\beta=\mathrm{const}\,(R_m-R_{m\,cr}),$$

i.e.,

$$|A|_{max}\sim\sqrt{R_m-R_{m\,cr}}. \tag{2.5}$$

Thus, for $R_m>R_{m\,cr}$ a nonstationary magnetic field develops having the frequencies $\Re e\,\omega$ and an amplitude proportional to $\sqrt{R_m-R_{m\,cr}}$. The velocity variation associated with this nonstationary field may be found by integrating the equation of motion which includes the Lorentz force. For a further increase of R_m the time at last arrives when the nonstationary motion $\mathbf{v}_0$, $\mathbf{H}_0$ which has been produced becomes unstable. In this case it follows that since the amplitude of the magnetic field is no longer small, one should use both Eq. (2.1) and the equation of motion. The velocity **v** and the magnetic field **H** are sought in the form

$$\mathbf{v}=\mathbf{v}_0+\mathbf{v}_1, \quad \mathbf{H}=\mathbf{H}_0+\mathbf{H}_1,$$

where $\mathbf{v}_1$ and $\mathbf{H}_1$ are small corrections which are now quantities of the same order of magnitude. Actually, the linearized equation of motion and induction equation have the form

$$\frac{\partial \mathbf{v}_1}{\partial t}+(\mathbf{v}_0\nabla)\mathbf{v}_1+(\mathbf{v}_1\nabla)\mathbf{v}_0=-\frac{\nabla p_1}{\varrho}+\nu\Delta\mathbf{v}_1$$
$$+\frac{1}{4\pi\varrho}[\mathbf{H}_0\operatorname{curl}\mathbf{H}_1]+\frac{1}{4\pi\varrho}[\mathbf{H}_1\operatorname{curl}\mathbf{H}_0], \quad \frac{\partial \mathbf{H}_1}{\partial t}=\operatorname{curl}[\mathbf{v}_0\mathbf{H}_1]+\operatorname{curl}[\mathbf{v}_1\mathbf{H}_0]+\nu_m\Delta\mathbf{H}_1, \tag{2.6}$$

and both of the quantities $\mathbf{v}_1$ and $\mathbf{H}_1$ are contained linearly in these equations. If the solution of Eq. (2.6) is now sought in the form

$$\mathbf{v}_1=\mathbf{f}_1 e^{-i\omega_1 t}, \quad \mathbf{H}_1=\mathbf{h}_1 e^{-i\omega_1 t},$$

we shall again arrive at the problem involving eigenvalues. However, now $\mathbf{f}_1$ and $\mathbf{h}_1$ are functions not only of just the coordinates but also time:

$$\mathbf{f}_1=\mathbf{f}_1(x, y, z, t), \quad \mathbf{h}_1=\mathbf{h}_1(x, y, z, t),$$

since the unperturbed motion is nonstationary. For the appearance of ω_1 with a positive imaginary part a new stationary motion is produced having the frequency $\alpha_1=\Re e\ \omega_1$, etc.

Actually, the case in which the motion is turbulent even before the onset of magnetic instability is of greater interest. Then the solutions of the induction equation should immediately be sought in the form

$$\mathbf{H}=\mathbf{h}(x, y, z, t)e^{-i\omega t},$$

since the unperturbed motion is now dependent on time. If for a certain R_m it turns out that $\Im m\ \omega>0$, then this is what will indicate the fact that turbulent motion generates a nondecaying magnetic field (i.e., a dynamo of small-scale fields holds).

Above, we have considered situations in which only R_m varied while R remained constant. For combined variation of R_m with R, intrinsically hydrodynamic instability may develop besides magnetic instability. The new nonstationary motion which develops as a result leads, in turn, to an additional variation of the magnetic field. From this the conclusion of the necessity of simultaneous solution of Eqs. (2.6) follows. The unknowns should be sought in the form $\boldsymbol{f}e^{-i\omega t}$, the preexponential factors depending both on the coordinates and on time.

If instead of time the phase $\varphi_i=\alpha_i t+\mu_i$ (for the velocity) and the phase $\psi_i=\delta_i t+\pi_i$ (for the magnetic field) are used as the independent variable, then the corresponding expressions must have the form (viz., for example, [3], where the expressions for the velocity are given)

$$\mathbf{v}(x, y, z, t)=\sum_{p_1, p_2, \ldots, p_n}\mathbf{A}_{p_1, p_2, \ldots, p_n}(x, y, z, t)\exp\left(-\sum_{i=1}^{n}p_i\varphi_i\right),$$
$$\mathbf{H}(x, y, z, t)=\sum_{p_1, p_2, \ldots, p_n}\mathbf{B}_{p_1, p_2, \ldots, p_n}(x, y, z, t)\exp\left(-\sum_{i=1}^{n}p_i\psi_i\right). \tag{2.7}$$

This is an expansion into a Fourier series in which not only terms with the fundamental frequencies α_i and δ_i are included but also the terms corresponding to frequencies equal to integer multiples of them (the summation of (2.7) is carried out over all possible steps of integer numbers $p_1, p_2, \ldots, p_n$).

It should be noted that starting with a certain i, the frequencies α and δ (as well as the initial phases μ and π) coincide with one another. However, the coincidence of all α and δ is not mandatory – one of the quantities (the velocity or the magnetic field) may correspond to a greater number of degrees of freedom than the other one. For example, if the onset of magnetic instability occurred first, when the motion was already turbulent, the velocity must correspond to a greater number of degrees of freedom. Coincidence between the frequencies and the initial phases develops from the instant when there are pulsations of both the velocity and the magnetic field which corresponds to simultaneous solution of the equations of motion and induction.

3. The equation for generation of a regular field (in the kinematic statement) would have to be derived from the induction equation by averaging in which the regular component of the magnetic field is isolated:

$$\frac{\partial \mathbf{H}}{\partial t}=\operatorname{curl}[\mathbf{vH}]+\nu_m\Delta\mathbf{H}, \quad \mathbf{H}=\mathbf{B}+\mathbf{h}, \quad \langle\mathbf{H}\rangle=\mathbf{B}, \tag{3.1}$$

where <...> denotes the averaging operation. In order to average the induction equation one should stipulate the turbulence model. In the case of gyrotropic turbulence for $R_m \ll 1$ Shteenbek and Krauze [5] derived the equation

$$\frac{\partial \mathbf{B}}{\partial t} = \alpha \operatorname{curl} \mathbf{B} + \nu_m \triangle \mathbf{B}. \tag{3.2}$$

For $R_m \gg 1$ the generation equation was derived by Vainshtein [6], the Gaussian turbulence model being used (i.e., the odd moments are equal to zero, while the even ones are expressed in terms of the second moments; viz. likewise [7]):

$$\frac{\partial \mathbf{B}}{\partial t} = \alpha \operatorname{curl} \mathbf{B} + (\nu_m + \nu_T) \triangle \mathbf{B}, \tag{3.3}$$

where ν_T is the turbulent viscosity.

These equations have undamped solutions (i.e., the gyrotropic turbulence generates a regular field).

Let us demonstrate that an equation more general than (3.3) may be derived without stipulating the specific turbulence model.

The time averaging of the derivative in the left side of the induction equation (3.1) and of the last term in the right side is trivial:

$$\left\langle \frac{\partial \mathbf{H}}{\partial t} \right\rangle = \frac{\partial \mathbf{B}}{\partial t}, \quad \langle \nu_m \triangle \mathbf{H} \rangle = \nu_m \triangle \mathbf{B},$$

so that the principal problem is the averaging of $[\mathbf{vH}]$. Since

$$\langle \operatorname{curl} [\mathbf{vH}] \rangle = \operatorname{curl} \langle [\mathbf{vH}] \rangle,$$

it is sufficient to average $[\mathbf{v}\,\mathbf{H}]$, conventional time averaging being used. As a result,

$$\langle [\mathbf{vH}] \rangle = \hat{L} \mathbf{B} \tag{3.4}$$

must be obtained, where L is a certain operator which is in general integral. Since $\mathbf{H}$ enters into $[\mathbf{vH}]$ linearly and the averaging operation is likewise linear, L must be a linear operator; i.e.,

$$\hat{L} \mathbf{B} = \int_0^\infty \hat{K}(t, t') \mathbf{B}(\mathbf{r}, t')\, dt', \tag{3.5}$$

where the kernel $\hat{K}$ may include differentiation with respect to the spatial coordinates. Let us clarify the form of $\hat{K}$.

First of all it is clear that as a result of averaging a vector quantity must be obtained. Therefore, K cannot contain differential operations which,being applied to $\mathbf{B}$, would yield a scalar. Further, since the chosen direction is absent in the space, one should choose isotropic operations. The sole differential operator which transforms a vector into a vector and does not contain the chosen directionis $\operatorname{curl}^n (n = 0, 1, \ldots)$.

Thus, $\hat{K}$ has the form

$$\hat{K} = \tilde{\alpha}_0(t, t') + \sum_{n=1}^{\infty} \tilde{\alpha}_n(t, t') \operatorname{curl}^n. \tag{3.6}$$

If we nowintroduce δ-correlation with respect to time (i.e., if it is assumed that the correlation time is short in comparison with the averaging period)

$$\tilde{\alpha}_m(t, t') = \alpha_m \delta(t - t') \quad (m = 0, 1, \ldots),$$

then

$$\langle \operatorname{curl} [\mathbf{vH}] \rangle = \sum_{n=1}^{\infty} \alpha_{n-1} \operatorname{curl}^n \mathbf{B}, \tag{3.7}$$

while the generation equation takes the form

$$\frac{\partial \mathbf{B}}{\partial t} = \sum_{n=1}^{\infty} \alpha_{n-1} \operatorname{curl}^n \mathbf{B} + \nu_m \triangle \mathbf{B}. \tag{3.8}$$

Limiting ourselves to the first two terms of the sum, we obtain the equation

$$\frac{\partial \mathbf{B}}{\partial t} = \alpha_0 \operatorname{curl} \mathbf{B} + (\alpha_1 + \nu_m) \triangle \mathbf{B}, \tag{3.8'}$$

which formally coincides with (3.3).

Since the magnetic field is an axial vector, the vectors $\alpha_0 \operatorname{curl} \mathbf{B}$ and $\alpha_1 \triangle \mathbf{B}$ must also be axial. But the rotor of an axial vector is a polar vector. Thus, in order for $\alpha_0 \operatorname{curl} \mathbf{B}$ to be an axial vector, α_0 must be a pseudoscalar. Evidently, α_1 is a true scalar (it may be shown that α_1 is the turbulent viscosity [8]).

If the turbulence model does not contain pseudoscalars (for example, if it is isotropic), then the generation equation can be simplified:

$$\frac{\partial \mathbf{B}}{\partial t} = (\alpha_1 + \gamma_m) \triangle \mathbf{B}. \tag{3.9}$$

This is the conventional diffusion equation with a diffusion coefficient $\alpha_1 + v_m$. Since in a real situation $\alpha_1 \gg \nu_m$, it follows from (3.9) that isotropic turbulence leads to anomalous damping of the field. In exactly the same way it may be shown that damping of the field takes place in the case of axisymmetric turbulence.

Thus, the derived general equation (3.8) shows that the generation condition is the presence of certain pseudoscalars in the turbulence model.

LITERATURE CITED

1. A. Schluter and L. Biermann, Ztschr. Naturforsch., 5a, 237 (1950).
2. L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Addison-Wesley (1959).
3. L. D. Landau, Dokl. Akad. Nauk SSSR, 44, 339 (1944).
4. G. K. Batchelor, Proc. Roy. Soc., 201A, 405 (1950).
5. M. Shteenbek and F. Krauze, Magnitn. Gidrodin., No. 3, 19 (1967).
6. I. Vainshtein, Zh. Éksp. Teor. Fiz., 58, No. 1, 153 (1970).
7. P. H. Roberts, Lectures in Appl. Math., 14, 129 (1971).
8. I. K. Chada, Acta Phys. Hungarica, No. 1, 235 (1952).