MAGNETOHYDRODYNAMIC SHEAR FLUID FLOWS WITH A RHEOLOGICAL POWER LAW UNDER CONDITIONS OF TRANSVERSE DRIFT

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Magnetohydrodynamic shear flows of a Newtonian dilatant fluid with a transverse velocity component are considered. Exact analytic solutions and numerical solutions are used to investigate the combined effect of the magnetic field and transverse drift of the medium on the velocity of the leading edge of the shear wave.

When studying the flow of a Newtonian fluid with a rheological power law, it was found that in dilatant media [1] shear disturbances propagate as waves whose leading edge travels with a finite velocity [2, 3]. In [2-5] the effect of different factors on the velocity of the leading edge of a shear wave was investigated. It was shown that an external transverse magnetic field reduces the velocity of the leading edge of a shear wave in media having conductivity and sometimes stops it completely [2, 5].

The presence of a transverse velocity component in a fluid stream, which leads to additional convective transfer of momentum, also affects the velocity of the leading edge [4]. When the direction of the transverse velocity component is opposite to the direction of propagation of the wave front, there is a similar stopping of the shear wave front, which leads to a spatial localization of the shear disturbances [4].

In the present paper we consider the combined effect of an external transverse magnetic field and transverse drift of the fluid in shear flows of conducting dilatant fluids.

Let an isotropically conducting incompressible dilatant fluid with density ρ and conductivity σ fill a space with a homogeneous external magnetic field $\mathbf{B}_0=(0,\ 0,\ B_0)=\text{const}$. An infinitely thin weightless porous plate through which the fluid moves with a constant velocity v_0 in the direction of the field $\mathbf{B}_0$ is placed in the plane $z=0$ in the fluid. Below, we consider the unsteady shear flows of a fluid caused by displacement of the plate in the direction of the x axis, which is perpendicular to the external field.

In the inductionless approximation ($\mathrm{Re}_m \ll 1$) an equation describing such shear flows for $\mathbf{E}=0$ can be written ($n>1$):

$$u_t=\mathfrak{L}u, \quad \mathfrak{L}=\alpha\frac{\partial}{\partial z}\left[\left|\frac{\partial}{\partial z}\right|^{n-1}\frac{\partial}{\partial z}\right]-\beta\frac{\partial}{\partial z}-\gamma. \tag{1}$$

Here, $u(z,\ t)$ is the x component of the fluid velocity, $\alpha=k/\rho$, $\beta=v_0=\text{const}$ is the transverse velocity component, $\gamma=\sigma B_0^2/\rho$, and k and n are rheological constants.

1. At the initial time $t=0$, let momentum $t=0$ referring to unit area be transferred to the plate. We find the velocity distribution $u(z,\ t)$ at an arbitrary time $t>0$. According to the formulation of the problem,

$$\begin{cases} u_t=\mathfrak{L}u, & -\infty<z<+\infty, \quad t>0, \\ u(z,0)=P\delta(z), \end{cases} \tag{2}$$

where $P=\mathfrak{P}/\rho$, and $\delta(z)$ is the Dirac delta function.

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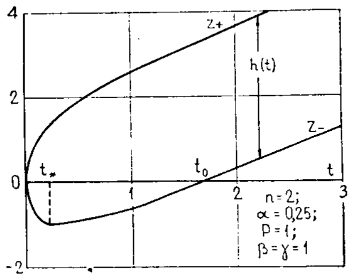

Fig. 1

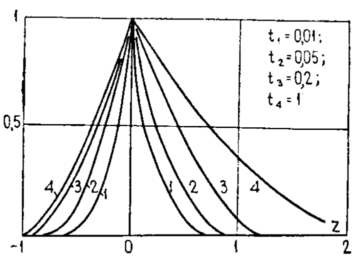

Fig. 2

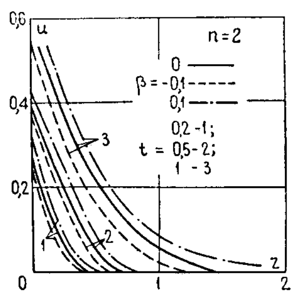

Fig. 3

Using the results of [2, 4], we replace the variables and the desired function

$$u(z,t)=w(z,t)\exp(-\gamma t),$$

$$\xi=z-\beta t,\quad \tau=\frac{1-\exp(1-n)\gamma t}{\gamma(n-1)}. \tag{3}$$

Problem (2) then transforms into the corresponding problem for the function $w(z, t)\equiv w(\xi, \tau)$

$$\begin{cases}\dfrac{\partial w}{\partial\tau}=\alpha\dfrac{\partial}{\partial\xi}\left[\left|\dfrac{\partial w}{\partial\xi}\right|^{n-1}\dfrac{\partial w}{\partial\xi}\right],\\ w(\xi,0)=P\delta(\xi).\end{cases} \tag{4}$$

It has been shown in [2] that problem (4) has the following solution:

$$w(\xi,\tau)=\begin{cases}\left(\dfrac{P^{n+1}}{\alpha\tau}\right)^{\frac{1}{2n}}\left[(2n)^{-\frac{1}{n}}\dfrac{n-1}{n+1}\left(\eta_0^{\frac{n+1}{n}}-|\eta|^{\frac{n+1}{n}}\right)\right]^{\frac{n}{n-1}} & \text{for } |\eta|<\eta_0;\\ 0 & \text{for } |\eta|\geqslant\eta_0;\end{cases} \tag{5}$$

$$\eta=(P^{n-1}\alpha\tau)^{-\frac{1}{2n}}\xi;$$

$$\eta_0=(2n)^{\frac{1}{2n}}\sqrt{\frac{n+1}{n-1}}\left[\frac{2n}{n+1}\mathrm{B}\left(\frac{n}{n+1},\frac{2n-1}{n-1}\right)\right]^{\frac{1-n}{2n}}.$$

With the substitutions (3), Eq. (5) determines the solution of the desired problem in the form of a shear wave in a fluid, and the position of the wave fronts $z_\pm(t)$ at an arbitrary time $t>0$ is determined by the dependence (Fig. 1)

$$z_\pm(t)=\pm\eta_0 P^{\frac{n-1}{2n}}\alpha^{\frac{1}{2n}}\left[\frac{1-\exp(1-n)\gamma t}{\gamma(n-1)}\right]^{\frac{1}{2n}}+\beta t. \tag{6}$$

Note that the front z_+ of the shear wave always propagates in the direction of transverse drift v_0 ($\dot z_+>0$ for $t>0$). The front z_- of the shear wave first moves opposite to the flow ($\dot z_-<0$) and its velocity decreases and, finally, at $t=t_*$ it stops (Fig. 1). Then the front changes direction ($\dot z_->0$ for $t>t_*$) and at some time $t=t_0$ it reaches the plate. The plate is now stopped, and the disturbance propagates in a fluid layer traveling in the direction v_0. It follows from (6) that the width of the region of shear disturbances $h(t)=z_+(t)-z_-(t)$ does not exceed at any time a certain value h_{max}, which depends on the induction of the external field, i.e.,

$$h(t)\leqslant h_{\max}=2\eta_0\left[\frac{P^{n-1}\alpha}{\gamma(n-1)}\right]^{\frac{1}{2n}}. \tag{7}$$

As the induction increases, h_{max} decreases.

The formulas obtained allow us to perform a limiting transition $\beta\to 0$ corresponding to the absence of a transverse velocity component [2] and a limiting transition $\gamma\to 0$ describing the flow of a nonconducting fluid.

2. We consider an unsteady shear flow, when for $t>0$ the plate moves with a constant velocity U_0. The corresponding boundary-value problem can be written

$$\begin{cases}u_t=\mathfrak{L}u, & -\infty<z<+\infty,\ t>0,\\ u(z,0)=0, & u(0,t)=U_0=\text{const}.\end{cases} \tag{8}$$

An exact self-similar solution cannot be obtained for (8). Therefore, the solution is obtained by the numerical method of finite differences, using an implicit three-layer conservative difference scheme for a straight-through calculation, which is obtained by the balance method

$$\frac{3}{2\tau}(u_{k,l+1}-u_{k,l})-\frac{1}{2\tau}(u_{k,l}-u_{k,l-1})$$

$$=\frac{\alpha}{h}\left[\left|\frac{u_{k+1}-u_k}{h}\right|^{n-1}\frac{u_{k+1}-u_k}{h}+\left|\frac{u_k-u_{k-1}}{h}\right|^{n-1}\frac{u_k-u_{k-1}}{h}\right]_{l+1}-\beta\frac{u_{k+1,l+1}-u_{k-1,l+1}}{2h}-\gamma u_{k,l+1}, \qquad (9)$$

where $u_{k,l}$ is the value of u at the node (kh, $l\tau$), h is a step in the spatial variable, and τ is a time step; for simplicity the square brackets are removed from the subscript $l+1$ in the right-hand side of (9). This scheme is certainly stable and has an error $O(h^2+\tau^2)$ [6].

The nonlinear system (9) is solved by the iterative method together with the sweep method, while it is linearized using the representation

$$\varphi^m \approx \bar{\varphi}^m + m\bar{\varphi}^{m-1}(\varphi-\bar{\varphi}),$$

where $\bar{\varphi}$ is the value of φ from the preceding iteration.

The condition for terminating the iterations at the layer is

$$\max_k |u_{k,l+1}-\bar{u}_{k,l+1}| \leqslant \varepsilon.$$

Figure 2 gives some results of numerical calculations of (8) for $n=3$; $\alpha=0.25$; $\beta=\gamma=1$; $U_0=1$.

As can be seen from the figure the solutions obtained have the form of shear waves whose fronts propagate along both sides of the plate – the source of the shear perturbations. The presence of transverse drift, which changes the front velocities, leads to an asymmetrical velocity distribution with respect to the plane $z=0$ (Fig. 2), which increases in time.

In the region $z<0$ where the shear-wave front moves in the direction opposite to the flow, the front stops at a finite distance from the plate. In the region $z>0$ the direction of the front coincides with the direction of the transverse drift and the front recedes to infinity. Analytic proof of the qualitative difference between the solutions of the problem for $z<0$ and $z>0$ is based on analysis of the stationary solution and is given in the Appendix.

3. Similarly, using the numerical methods mentioned in Sec. 2, we have for the solution of the boundary-value problem

$$\begin{cases} u_t=\mathfrak{L}u, & 0<z<\infty,\ t>0, \\ u(z,0)=0, & \left(\dfrac{\partial u}{\partial z}\right)\Big|_{z=0}=-T=\text{const}. \end{cases} \qquad (10)$$

The results of numerical calculations of the velocity distribution of the fluid for $\alpha=0.25$, $\gamma=T=1$ are given in Fig. 3 for different values of β. It can be seen that the magnitude and direction of the transverse velocity component of the fluid affects the velocity of the shear wave front.

APPENDIX

The solution of the evolutionary problem (8) for $t\to\infty$ has as a limit the solution of the stationary problem ($\beta>0$, $\gamma>0$)

$$\begin{cases} \alpha\dfrac{d}{dz}\left(\left|\dfrac{du}{dz}\right|^{n-1}\dfrac{du}{dz}\right)-\beta\dfrac{du}{dz}-\gamma u=0, & -\infty<z<\infty, \\ u(0)=1, \end{cases} \qquad (11)$$

which is bounded at infinity (for simplicity, we assume that $U_0=1$).

Therefore, if the shear wave front stops at a finite distance from the plate ($z=0$), the solution of the stationary problem must vanish together with the first derivative at some point $z=z_0$, and $|z_0|<\infty$. If the solution of the stationary problem (11) does not vanish at a finite distance from the plate, then the shear wave front described by the nonstationary problem (8) recedes to infinity.

We will show that the solution of (11) vanishes at a finite distance only in the region $z<0$ (where the shear wave front propagates opposite to the moving stream), and in the region $z>0$ there is no such point.

Actually, in the region $z<0$, $du/dz>0$; therefore, setting $du/dz=\psi^{1/(n+1)}$ in (11), we can obtain

$$\begin{cases} \dfrac{d\psi}{du}=\dfrac{n+1}{\alpha n}\left(\gamma u+\beta\psi^{\frac{1}{n+1}}\right), \quad 0\leqslant u\leqslant 1; \\ \psi(0)=0. \end{cases} \tag{12}$$

According to the theorem on the estimates of solutions of the differential equation (12), we have [7]

$$\psi\geqslant\frac{n+1}{2\alpha n}\gamma u^2, \text{ and } u\leqslant\left[1+\frac{n-1}{n+1}\left(\frac{n+1}{2n}\frac{\gamma}{\alpha}\right)^{\frac{1}{n+1}}z\right]^{\frac{n+1}{n-1}}. \tag{13}$$

Consequently, in the region $z<0$ the solution of (11) vanishes at some point $z=z_0$, and

$$z_0\geqslant-\frac{n+1}{n-1}\left(\frac{2n}{n+1}\frac{\alpha}{\gamma}\right)^{\frac{1}{n+1}}. \tag{14}$$

Similarly, in the region $z>0$, $du/dz<0$ and with the substitution $du/dz=-f^{1/(n+1)}$ problem (11) transforms into

$$\begin{cases} \dfrac{df}{du}=\dfrac{n+1}{\alpha n}\left(\gamma u-\beta f^{\frac{1}{n+1}}\right), \quad 0\leqslant u\leqslant 1; \\ f(0)=0. \end{cases} \tag{15}$$

From this

$$f\leqslant\left(\frac{\gamma}{\beta}u\right)^{n+1}, \text{ and } u\geqslant\exp\left(-\frac{\gamma z}{\beta}\right),$$

and in the region $z>0$ the solution of (11) vanishes only at an infinitely distant point.

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