STEADY-STATE MAGNETO- AND ELECTROHYDRODYNAMIC FLOWS IN A NON-NEWTONIAN FLUID

L. K. Martinson

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Steady-state gradient and shear MHD and EHD flows in flat channels are discussed for non-Newtonian fluids obeying the rheologic law proposed by V. V. Novozhilov. The solutions for the fluid velocity distributions in the channels are reduced to quadratures. The effect of the various problem parameters on the nature of the flows is analyzed. The limiting transitions to flows of a Newtonian fluid and to hydrodynamic flows are discussed.

In a study of MHD and EHD flows of non-Newtonian fluids, one uses most often a model of a medium obeying the rheologic power law

$$S_{ij}=2kI^{n-1}f_{ij},\quad n>0. \tag{1}$$

Here $I = \sqrt{2f_{ij}f_{ij}}$ is the invariant of the tensor of the strain rates f_{ij}, k is the rheologic consistency parameter, and n is the index of the non-Newtonian state.

As was pointed out [1], however, there is interest in an expansion of the classes of non-Newtonian media and in the introduction of new rheologic models in theoretical studies of magneto- and electrohydrodynamics.

The motion of media obeying the rheologic law introduced in [2] are discussed below. For these media, the relation between the deviator of the tensor of the stresses S_{ij} and the tensor of the strain rates f_{ij} for flows in channels with fixed walls has the form

$$S_{ij}=2\mu bT^{n}f_{ij},\qquad T=\frac{u_k u_k}{\nu I}, \tag{2}$$

where u_k is the velocity vector of the fluid: μ and ν are the dynamic and kinematic coefficients of viscosity; T is a dimensionless invariant which depends at each point in the fluid medium on the kinetic energy of the flow and on the invariant of the strain rate tensor; b and n are certain constants in the theory with $0 \le n < 1$ and $b > 0$. When $n = 0$ and $b = 1$, Eq. (2) transforms into the usual rheologic law for a viscous Newtonian fluid.

1. Magnetohydrodynamic Flows

Hartmann Flow. We consider MHD flow of a conducting medium obeying the rheologic law (2) in a flat channel $\overset{-}{}L \le x \le \overset{+}{}L$ under the influence of a constant pressure gradient $P = -\partial P/\partial z = \text{const} > 0$ in the presence of a uniform electric field $E_y = E_0$. Furthermore, a magnetic field $B_x = B_0$ is perpendicular to the direction of motion of the fluid.

For plane-parallel flows, Eq. (2) yields an expression for the shear stress in the form

$$\tau=\mu b\left[u^2\nu^{-1}\left|\frac{du}{dx}\right|^{-1}\right]^{n}\frac{du}{dx}=\mu_*\frac{du}{dx}, \tag{3}$$

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with the effective viscosity μ_* depending not only on the velocity gradient, but also on the fluid velocity itself at the point of flow under consideration. If U and L are characteristic velocities and lengths, the characteristic value of the effective viscosity in the flow,

$$\mu_0=\mu b\left(\frac{UL}{\nu}\right)^n=\mu b\,\mathrm{Re}^n, \tag{4}$$

increases as the Reynolds number increases when $n \neq 0$.

A dimensionless equation describing steady-state plane-parallel MHD flow of such a fluid in the channel can be written in the form

$$\mathrm{D}[v^{2n}|\mathrm{D}v|^{1-n}\,\mathrm{sgn}\,\mathrm{D}v]-\mathrm{M}^2v=-1, \tag{5}$$

where

$$\mathrm{D}\equiv\frac{\mathrm{d}}{\mathrm{d}\xi};\quad \xi=\frac{x}{L};\quad v=\frac{u}{U};$$

$$U=(P+\sigma E_0B_0)\frac{L^2}{\mu_0};\quad \mathrm{M}^2=L^2B_0^2\frac{\sigma}{\mu_0}.$$

The dimensionless parameter M can be called the generalized Hartmann number, since it transforms into the ordinary Hartmann number when n = 0 and b = 1. We point out that there is a relation between the ordinary Hartmann number Ha and the generalized Hartmann number M introduced in this theory:

$$\mathrm{M}^2=\mathrm{Ha}^2/b\,\mathrm{Re}^n. \tag{6}$$

Equation (5) should be solved under the boundary condition of fluid adhesion,

$$v(1)=v(-1)=0. \tag{7}$$

Taking into consideration the symmetry of flow with respect to the channel center at $\xi = 0$, one can replace one of the conditions (7) with the condition $\mathrm{D}v = 0$ at $\xi = 0$ and consider only the flow in half the channel. In the following, we limit our consideration to the flow region $-1 < \xi < \mathrm{D}$, where $\mathrm{D}v > 0$.

At the center of the channel ($\xi = 0$), let the fluid velocity reach the value $v_{max} = \alpha$ and let the average velocity in the channel be $v_{av} = \beta$. It is obvious that $\beta \le \alpha \le \alpha_0$, where $\alpha_0 = \mathrm{M}^{-2}$. The quantity α_0 corresponds to the flow velocity in a nonviscous fluid and, therefore, the velocity of a fluid possessing viscosity cannot exceed this value.

By introducing the function $F = v(1 + n)/(1-n)$ and integrating once with the boundary condition at the center of the channel taken into account, Eq. (5) can be transformed to

$$\frac{\mathrm{d}f}{\mathrm{d}\xi}=N\left\{(\theta-f)-\mathrm{M}^2\frac{1+n}{2}\left(\theta^{\frac{2}{1+n}}-f^{\frac{2}{1+n}}\right)\right\}^{\frac{1}{2-n}}, \tag{8}$$

where

$$N=\left\{\frac{2-n}{1-n}\left(\frac{1+n}{1-n}\right)^{1-n}\right\}^{\frac{1}{2-n}},\quad \theta=\alpha^{\frac{1+n}{1-n}},\quad \text{with}\quad 0\le\theta\le\mathrm{M}^{\frac{2(1+n)}{n-1}}.$$

Integrating Eq. (6) with respect to ξ between the limits -1 and 0, we obtain

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\xi}\right|_{\xi=-1}=\frac{1+n}{1-n}\,(1-\mathrm{M}^2\beta)^{\frac{1}{1-n}}. \tag{9}$$

Taking Eq. (9) into consideration, Eq. (8) makes it possible to find a relation between the average flow velocity β and the maximum flow velocity α in the form

$$\mathrm{M}^2\beta=1-\left\{\frac{2-n}{1+n}\left(\alpha^{\frac{1+n}{1-n}}-\mathrm{M}^2\frac{1+n}{2}\,\alpha^{\frac{2}{1-n}}\right)\right\}^{\frac{1-n}{2-n}}. \tag{10}$$

Integrating Eq. (8), we obtained a relationship $\xi = \xi(f)$ in the form of a quadrature

$$\xi = \frac{1}{N}\int_0^f \left[(\theta-\psi) - M^2 \frac{1+n}{2}\left(\theta^{\frac{2}{1+n}} - \psi^{\frac{2}{1+n}}\right)\right]^{\frac{1}{n-2}} d\psi - 1 \tag{11}$$

and taking the boundary condition f(0) = into account, we obtain the following equation for the determination of θ:

$$\int_0^\theta \left[(\theta-\psi) - M^2 \frac{1+n}{2}\left(\theta^{\frac{2}{1+n}} - \psi^{\frac{2}{1+n}}\right)\right]^{\frac{1}{n-2}} d\psi = N. \tag{12}$$

In the neighborhood of the channel wall (in the neighborhood of the point $\xi = -1$), Eq. (11) can be written in the form

$$f(s) \approx N\left[\theta - \frac{1+n}{2} M^2 \theta^{\frac{2}{1+n}}\right]^{\frac{1}{2-n}} s \tag{13}$$

by introducing the variable $s = 1 + \xi$, which characterizes the distance measured from the wall.

The quadratures in (11) and (12) are expressed through elementary functions for certain values of the parameters. In particular, when M = 0, we have from Eqs. (11) and (12)

$$f(\xi) = \theta\left[1 - (-\xi)^{\frac{2-n}{1-n}}\right], \quad \theta = \frac{1+n}{2-n}. \tag{14}$$

This case corresponds to the flow of a nonconducting fluid discussed in [2]. The velocity distribution of the fluid in the channel is shown in Fig. 1 in accordance with Eq. (14) for various values of the parameter n. As is clear from the figure, the velocity profile is flattened as n increases.

When n = 0 and b = 1, Eqs. (11) and (12) define the classical Hartmann profile (for this case, M = Ha),

$$f(\xi) = \theta \frac{\mathrm{ch\,Ha} - \mathrm{ch\,Ha}\xi}{\mathrm{ch\,Ha} - 1}, \quad \theta = \frac{\mathrm{ch\,Ha} - 1}{\mathrm{Ha}^2\,\mathrm{ch\,Ha}}. \tag{15}$$

In the general case, calculation of the fluid velocity distribution in the channel can be carried out in the following manner. We transform Eq. (12) into a form convenient for calculation,

$$\int_0^1 \left[(1-\psi) - C\left(1-\psi^{\frac{2}{1+n}}\right)\right]^{\frac{1}{n-2}} d\psi = N\theta^{\frac{n-1}{2-n}}. \tag{16}$$

Here

$$C = \frac{1+n}{2} M^2 \theta^{\frac{1-n}{1+n}}, \quad 0 \leqslant C \leqslant \frac{1+n}{2}. \tag{17}$$

We determine θ by assigning values to n and C and computing the quadrature in (16); we determine the corresponding value of M^2 from Eq. (17). This makes it possible to construct the relationship $\theta = \theta(M^2, n)$. The nature of this relationship for n = 0.75 is shown in Fig. 2. Equation (16) indicates that the quantity θ decreases as the generalized Hartmann number M increases with $\theta < M^{2(1+n)/(1-n)}$ for any finite value of $M[C < 1/_2\,(1+n)]$, since the integrand in Eq. (16) has an integrable singularity at the point $\psi = 1$. When $M \to \infty$, $\theta \to 0$ and $M^2\theta^{(1-n)/(1+n)} \to 1$. It then follows that the formation of a quasisolid core in the center of the channel, which was observed in [3] for MHD flows obeying the rheological law (1), does not occur for this rheological model (2).

After the value of θ has been determined, Eq. (11) yields the relationship $\xi = \xi(f)$ which determines the desired velocity distribution $v = v(\xi)$. The dependence of f/θ on $s = 1 + \xi$ for n = 0.75 is shown in Fig. 3 for various values of the generalized Hartmann number. Shown in the same figure for comparison by the dashed line is the parabolic Poiseuille profile for n = 0 and M = 0. As is clear from the figure, the velocity profile flattens as M increases so that the entire change in the velocity occurs in a very narrow boundary layer close to the wall for large values of M.

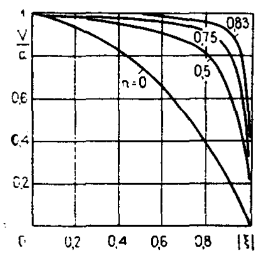

Fig. 1

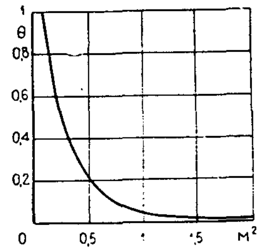

Fig. 2

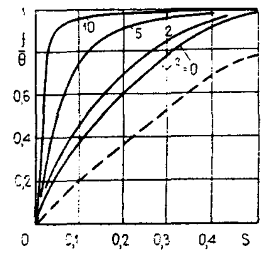

Fig. 3

To construct the velocity distribution of the fluid in this boundary layer, one can use an asymptotic representation of Eq. (13), which can be transformed for large values of M to the form

$$\frac{v}{a} \approx R^{\frac{1-n}{1+n}} M^{\frac{2(1-n)}{2-n}} s^{\frac{1-n}{1+n}}, \quad R = N\left(\frac{1-n}{2}\right)^{\frac{1}{2-n}}. \tag{18}$$

For n = 0.75, $v \sim s^{1/7}$ in the boundary-layer region.

Couette Flow. For uniform translational motion of the bounding surfaces (channel walls), the rheologic law (2) should be changed [2] so that the velocity U_k of the boundary appears in the invariant T. For this case [2]

$$T = \frac{(U_k - u_k)(U_k - u_k)}{\nu l}. \tag{19}$$

We consider steady-state MHD Couette flow of such a non-Newtonian fluid in a flat channel with an external uniform transverse magnetic field $B_x = B_0$ = const. The nonconducting walls of the channel at $x = \pm L$ then move in opposite directions at the identical velocity V = const and there is no electric field in the channel.

Taking Eqs. (2) and (19) into consideration, the dimensionless equation describing such shear flow in the region $0 \leq x \leq L$ has the form

$$\begin{gathered} \mathrm{D}[(1-v)^{2n}(\mathrm{D}v)^{1-n}] - \mathrm{M}^2 v = 0; \\ \mathrm{D} \equiv \frac{d}{d\xi}, \quad \xi = \frac{x}{L}, \quad v = \frac{u}{V}, \quad \mathrm{M}^2 = \frac{\mathrm{Ha}^2}{b\,\mathrm{Re}^n}, \quad \mathrm{Ha} = B_0 L\sqrt{\sigma/\mu}, \quad \mathrm{Re} = \frac{VL}{\nu}. \end{gathered} \tag{20}$$

Equation (20) must be solved under the obvious boundary conditions v(0) = 0 and v(1) = 1. Making the substitution

$$v = 1 - w^{\frac{1-n}{1+n}}, \tag{21}$$

we obtain the following nonlinear boundary-value problem (Dw < 0):

$$\begin{gathered} \mathrm{D}[(-\mathrm{D}w)^{1-n}] + \mathrm{M}^2\left(\frac{1+n}{1-n}\right)^{1-n}\left(w^{\frac{1-n}{1+n}} - 1\right) = 0; \\ w(0) = 1, \quad w(1) = 0 \end{gathered} \tag{22}$$

for the determination of the function w(ξ).

Integrating Eq. (22) once, we obtain

$$\begin{gathered} \mathrm{D}w = -\left[A - \mathrm{M}^2 G\left(w - \frac{1+n}{2} w^{\frac{2}{1+n}}\right)\right]^{\frac{1}{2-n}}; \\ G = \frac{2-n}{1-n}\left(\frac{1+n}{1-n}\right)^{1-n}. \end{gathered} \tag{23}$$

Here A > 0 is a constant of integration.

Equation (23) makes it possible to write the solution of the boundary-value problem (22) in the form of the quadrature

$$\int_0^w \left[A - M^2 G\left(z - \frac{1+n}{2} z^{\frac{2}{1+n}}\right)\right]^{\frac{1}{n-2}} dz = 1 - \xi, \tag{24}$$

where the constant $A = A(M^2, n)$ should be found from the condition

$$\int_0^1 \left[A - M^2 G\left(z - \frac{1+n}{2} z^{\frac{2}{1+n}}\right)\right]^{\frac{1}{n-2}} dz = 1. \tag{25}$$

An analysis of Eq. (25) indicates that

$$A \geqslant A_+ = M^2 G \frac{1-n}{2}, \tag{26}$$

and the value $A = A_+$ is reached when $M^2 \to \infty$. Therefore, for sufficiently large values of the generalized Hartmann number the relationship $A = A(M^2, n)$ can be approximated by the simple formula

$$A \approx M^2 G \frac{1-n}{2}. \tag{25}$$

If the value of A is found from Eq. (25), Eq. (24) yields the implicit relationship $\xi = \xi(w)$ which determines the desired velocity distribution $v = v(\xi)$.

The velocity distributions $v(\xi)$ for various values of the generalized Hartmann number are shown in Figs. 4 and 5 for n = 0.5 and n = 0.75. The dashed lines in these figures correspond to the linear velocity profile in the h. codynamic Couette flow of a Newtonian fluid.

Note that the following asymptotic expressions can be obtained from Eq. (24):

$$\begin{aligned} v &\approx 1 - A^{\frac{1-n}{1+n}} (1-\xi)^{\frac{1-n}{1+n}} \quad \text{near the point} \quad \xi = 1; \\ v &\approx \frac{1-n}{1+n}\left[A - M^2 G \frac{1-n}{2}\right]^{\frac{1}{2-n}} \xi \quad \text{near the point} \quad \xi = 0. \end{aligned} \tag{27}$$

The two limiting transitions $M \to 0$ and $n \to 0$ can be carried out in Eqs. (24) and (25). In these cases, the quadratures in (24) and (25) are expressed in terms of elementary functions.

Setting M = 0 in Eqs. (24) and (25), we have A = 1, $w = 1 - \xi$, and, finally,

$$v(\xi) = 1 - (1-\xi)^{\frac{1-n}{1+n}}.$$

This case corresponds to the shear flow of a nonconducting fluid considered in [2].

When n = 0 and b = 1 (M = Ha, v = 1−w), we obtain from Eqs. (24) and (25) the well-known velocity distribution in laminar MHD Couette flow for a Newtonian viscous fluid

$$v(\xi) = \frac{\operatorname{sh} \mathrm{Ha}\, \xi}{\operatorname{sh} \mathrm{Ha}}, \quad A = \mathrm{Ha}^2 \operatorname{cth}^2 \mathrm{Ha}.$$

2. Electrohydrodynamic Flow

We consider steady-state EHD flow of an incompressible spatially charged fluid obeying the rheologic law (2) in a flat channel $-L \le x \le +L$ with dielectric walls under the influence of a constant pressure gradient $P = -\partial p/\partial z$ = const in a uniform external electric field $E_z = E_0$ = const directed along the axis of the channel. The charge q_0 per unit length of the channel is one of the known parameters of the problem.

We select as characteristic quantities the channel half-width L and the charge density $\rho_0 = q_0/2L$. Then the distribution of the dimensionless charge density $\rho(\xi)$, which is determined from the absence of a transverse component of the current density, can be written in the form [5]

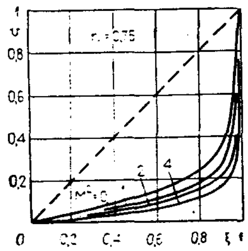

Fig. 4

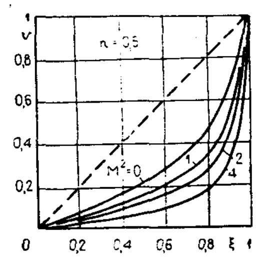

Fig. 5

$$\varrho(\xi)=\frac{\gamma}{\operatorname{tg}\gamma}\frac{1}{\cos^2\gamma\xi},\quad -1\leqslant\xi=\frac{x}{L}\leqslant+1. \tag{28}$$

The quantity $\gamma < {}^1/_2\,\pi$ in Eq. (28) is defined as the solution of the transcendental equation

$$\gamma\operatorname{tg}\gamma=\frac{aq_0L}{4\varepsilon\varepsilon_0 D}.$$

Here a is the charge mobility, D is the diffusion coefficient of the charged particles, ε is the dielectric constant of the fluid, and ε_0 is the dielectric constant of a vacuum.

The charge distribution (28) in the fluid is nonuniform with the maximum charge density being reached near the channel walls. Therefore, additional electrodynamic forces arise in the external electric field in the fluid with these forces being significant in layers near the walls and producing a considerable change in the velocity profile of the fluid in the channel.

Taking the quantity

$$u_0=\left\{\frac{PL^{2-n}\gamma^n}{\mu b}\right\}^{\frac{1}{n+1}}$$

as a characteristic velocity for the problem, we write the dimensionless equation of motion for this flow:

$$\mathrm{D}[\varepsilon^{2n}|\mathrm{D}v|^{1-n}\operatorname{sgn}\mathrm{D}v]+1+\frac{\gamma}{\operatorname{tg}\gamma}\frac{\Pi}{\cos^2\gamma\xi}=0;$$
$$\mathrm{D}\equiv\frac{\mathrm{d}}{\mathrm{d}\xi},\quad \xi=\frac{x}{L},\quad v=\frac{u}{u_0},\quad \Pi=\frac{q_0E_0}{2LP}, \tag{29}$$

having taken Eqs. (2) and (28) into consideration. The electrical interaction parameter π characterizes the ratio between electric forces and pressure forces.

Integrating Eq. (29) once with flow asymmetry with respect to the center of the channel taken into consideration, we obtain the following nonlinear differential equation for determination of the velocity profile:

$$\varepsilon^{2n}|\mathrm{D}v|^{1-n}\operatorname{sgn}\mathrm{D}v+\xi+\Pi\frac{\operatorname{tg}\gamma\xi}{\operatorname{tg}\gamma}=0. \tag{30}$$

If $E_0 > 0$ $(\pi \geq 0)$, $\pi > 0$ for this flow in all flow regions and $Dv < 0$ in the upper half of the channel. In this case, by introducing the function $\varphi = v^{1+n/1-n}$, Eq. (30) in the region $0 \leq \xi \leq 1$ can be written in the form

$$\frac{d\varphi}{d\xi}=-\frac{1+n}{1-n}\left(\xi+\Pi\frac{\operatorname{tg}\gamma\xi}{\operatorname{tg}\gamma}\right)^{\frac{1}{1-n}}. \tag{31}$$

Integrating Eq. (31) with fluid adhesion at the boundary taken into consideration, $\varphi(1) = 0$, we obtain

$$\varphi(\xi)=\frac{1+n}{1-n}\int_{\xi}^{1}\left(\eta+\Pi\frac{\operatorname{tg}\gamma\eta}{\operatorname{tg}\gamma}\right)^{\frac{1}{1-n}}d\eta \tag{32}$$

and, finally, for the fluid velocity distribution in the channel

$$v(\xi)=\left\{\frac{1+n}{1-n}\int_{\xi}^{1}\left(\eta+\Pi\frac{\operatorname{tg}\gamma\eta}{\operatorname{tg}\gamma}\right)^{\frac{1}{1-n}}d\eta\right\}^{\frac{1-n}{1+n}}. \tag{33}$$

The quadrature in Eq. (33) can be expressed through elementary functions for certain values of the rheologic parameter n. In particular, we have for n = 0.5 [6]:

$$I(\eta)=\int\left(\eta+\Pi\frac{\operatorname{tg}\gamma\eta}{\operatorname{tg}\gamma}\right)^2 d\eta=\frac{\eta^3}{3}+\frac{\Pi^2}{\gamma\operatorname{tg}^2\gamma}(\operatorname{tg}\gamma\eta-\gamma\eta)+\frac{2\Pi}{\gamma^2\operatorname{tg}\gamma}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2^{2k}(2^{k-1}-1)}{(2k+1)(2k)!}B_{2k}\gamma^{2k-1}\eta^{2k+1},$$

where B_n is the Bernoulli number.

The limiting transition n → 0 when b = 1 makes it possible to obtain from Eq. (33) the velocity distribution in EHD flow of a viscous Newtonian fluid [5]:

$$v(\xi)=\frac{1-\xi^2}{2}+\frac{\Pi}{\gamma\operatorname{tg}\gamma}\ln\frac{\cos\gamma\xi}{\cos\gamma}.$$

The other limiting case Π = 0 corresponds to the flow of an uncharged fluid. Equation (33) for Π = 0 obviously leads to Eq. (14), which was obtained previously in the study of MHD Hartmann flow.

We note in conclusion that the rheologic law (2) was introduced in [2] to describe average flow in steady-state turbulent flows of an incompressible viscous fluid.

Such an approach to the description of steady-state turbulent flows of fluids in channels as flows of a fluid obeying some rheologic law open up new applications for the magnetohydrodynamics of non-Newtonian fluids.

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