MOTION OF BOUNDARY OF SUPPORT OF GENERALIZED SOLUTION IN MAGNETIC RHEOLOGY PROBLEMS

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We investigate the manner in which shear perturbations propagate in magnetized dilatational media for processes described by the first boundary-value problem for the equation of magnetohydrodynamics of non-Newtonian fluids [1] in the region $R_+^2=\{(t,z):t\geqslant 0,\ 0\leqslant z<+\infty\}$:

$$u_t=(|u_z|^{n-1}u_z)_z-\gamma u,\quad t>0,\quad z\in R_+^1;$$
$$u(0,z)=u_0(z),\quad u(t,0)=u_1(t),\quad u(t,\infty)=0. \tag{1}$$

Here $n>1$, γ = const $\geqslant 0$, $u_0(z)$ is a nonnegative finite function with a bounded support supp $u_0(z)$ = $[0,\ l_0]$, $u_1(t)\geqslant 0$ for all $t\in[0,\infty)$, where $u_1(0)$ = $u_0(0)$.

Physically, this problem corresponds to the propagation of shear perturbations in a magnetized dilatational conducting fluid occupying the half-space $z>0$. Shear flow is induced for $t>0$ by prescribed motion of a weightless plate (the source of the shear perturbations) lying on the surface of the fluid. The velocity distribution at the initial moment of time t = 0 is localized near the plate in a region whose dimension is determined by the dimension l_0 of the support of the initial distribution.

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Singularities in the propagation of shear perturbations in dilatational media ($n>1$) come about because the effective viscosity of the medium reduces to zero for $u_z = 0$, as a result of which the velocity of propagation of the perturbations proves to be finite [1]. Mathematically, a finite velocity of propagation of shear perturbations in dilatational media means that for $n>1$ the carrier of the solution of problem (1) supp u remains bounded at any finite moment of time. In view of the degeneracy of Eqs. (1) for $u_z = 0$, we shall understand the solution of the boundary-value problem to mean, in general, the solution in the generalized sense [2, 3].

In a dilatational medium, a shear perturbation propagates from the source over the null unperturbed background in the form of a shear wave. At the point $z = z_+(t)$ of the shear-wave front which, at a given moment of time, separates the unperturbed region (where $u = 0$) from the region already reached by the shear wave, the physical conditions of continuity of velocity and tangential stress must be satisfied. These conditions require that $u\in C^1(R_+^1)$, i.e., in particular, that $u\to 0$ and $u_z\to 0$ as $z\to z_+ + (t) - 0$. The front of a shear wave is thus always slightly sloping.

We investigate the motion of the frontal point of a shear wave as a function of the motion of the plate and the initial velocity profile in the fluid.

THEOREM 1. The dimension of the support of the solution of problem (1) does not diminish with time.

Proof. The substitution $u(t, z) = w(t, z)\exp[-\gamma^t]$ reduces problem (1) to the following boundary-value problem in the half-strip $\Pi=\{(\tau,z):0\leqslant\tau<\tau_0, z\in R_+^1\}$ for the function $w(\tau, z) = w(\tau(t), z)$:

$$\begin{cases} w_\tau=(|w_z|^{n-1}w_z)_z, \\ w(0,z)=w_0(z)\equiv u_0(z), \quad w(\tau,0)=w_1(\tau), \quad w(\tau,\infty)=0. \end{cases} \tag{2}$$

Here the variables τ and t are related by

$$\tau=\tau_0\{1-\exp[-\gamma(n-1)t]\}, \quad \tau_0=[\gamma(n-1)]^{-1}. \tag{3}$$

We note that by virtue of the connection between $u(t, z)$ and $w(\tau, z)$, the position of the frontal point $z = z_+(t)$ for problem (1) at any moment of time is uniquely given by the position of the frontal point $z = z_+(\tau)$ for problem (2) through relationship (3), which connects t and τ.

Differentiating the obvious identity $w[\tau, z_+(\tau)]\equiv 0$ with respect to τ, we obtain [using (2)] the following expression for the velocity of motion of the frontal point in problem (2):

$$\frac{dz_+(\tau)}{d\tau}=-\lim_{z\to z_+-0}\frac{|w_z|^{n-1}w_z}{w}. \tag{4}$$

Since in the unperturbed region near the front $w>0$ and $w_z<0$ accordingly, by (4), $dz_+(\tau)/d\tau\geqslant 0$. Thus,

$$\dot z_+(t)=\frac{dz_+(t)}{dt}=\frac{dz_+(\tau)}{d\tau}\frac{d\tau}{dt}. \tag{5}$$

Since $d\tau/dt\geqslant 0$ for all $t>0$, we have from (5) that $\dot z_+(t)\geqslant 0$.

In this manner, the direction of motion of the boundary of the support of the solution of problem (1) does not change in time; accordingly, the dimension of the support of the solution or problem (1) does not diminish with time.

Remark. Theorem 1 also holds for the case when $\gamma = \gamma(t)$, where $\gamma(t)$ is an arbitrary nonnegative function of time.

Depending on how the support boundary $z = z_+(t)$ moves, we shall distinguish the following possible modes of shear flow.

1. If $z_+(t)\to\infty$ as $t\to\infty$, then this flow mode will be said to be nonlocalized.

2. If $z_+(t)\leqslant L<\infty$ for all $t\in[0,\infty)$, then this flow mode will be said to be spatially localized. If also $\dot z_+(t) = 0$ for all $t\in[0,T]$, where $T<+\infty$, the mode of localization will, in accordance with [4], be said to be metastable, while if $\dot z_+(t)\equiv 0$ for all $t\in[0,\infty)$, it will be said to be stable. Besides these modes, a mode of spatial localization is also possible:

when $\dot{z}_+(t)>0$ at any fixed moment of time and $\dot{z}_+(t) \to 0$ as $t \to \infty$.

We note that the absence of metastable localization is concomitant with the absence of stable localization of shear perturbations.

THEOREM 2. Let $u(t, z)$ be the solution of problem (1) and let $u(0, z) = u_0(z) \leqslant A(1-z/l_0)^{\frac{n+1}{n-1}}$ for $z \leqslant l_0$ and $u(t, 0) = u_1(t) \leqslant A$ for all $t \in [0, \infty)$, where l_0 is the dimension of the support of the initial distribution $[z_+(0) = l_0]$ and the constant A is connected with l_0 by

$$A = \left\{ \frac{\gamma(n-1)}{2n} l_0^{n+1} \left(\frac{n-1}{n+1} \right)^n \right\}^{\frac{1}{n-1}}. \tag{6}$$

Then the solution of problem (1) is stably localized.

Proof. Consider the function

$$v(t, z) \equiv v_0(z) = \begin{cases} A(1-z/l_0)^{\frac{n+1}{n-1}} & \text{for} \quad z < l_0, \\ 0 & \text{for} \quad z \geqslant l_0, \end{cases} \tag{7}$$

which is the solution in R_+^2 of the boundary-value problem

$$v_t = (|v_z|^{n-1} v_z)_z - \gamma v, \quad v(0, z) = v_0(z), \quad v(t, 0) = v_1(t) = A, \quad v(t, \infty) = 0. \tag{8}$$

Since $u_0(z) \leqslant v_0(z)$ and $u_1(t) \leqslant v_1(t)$, the monotonic dependence of the solution of problems (1) and (8) on the initial and boundary conditions [3] implies that $u(t, z) \leqslant v(t, z)$ everywhere in R_+^2. However, since $v(t, z) = 0$ for $z \geqslant l_0$ and for any $t \in [0, \infty)$, we have that $u(t, z) = 0$ for $z \geqslant l_0$ and for all $t \geqslant 0$. Consequently, $z_+(t) \leqslant l_0$. On the other hand, $z_+(0) = l_0$ and by virtue of Theorem 1, $\dot{z}_+(t) \geqslant 0$ for all $t > 0$. Hence, it follows that $z_+(t) \equiv l_0$ for all $t \in [0, \infty)$. Thus supp $u(t, z) = [0, l_0]$ for all $t \in [0, \infty)$. The theorem is proved.

To illustrate the possibility of realizing the mode of stable localization, we consider the exact solution of problem (1) with corresponding initial and boundary conditions satisfying the requirements of Theorem 2:

$$u(t, z) = \begin{cases} A\{1 + P\exp[\gamma(n-1)t]\}^{\frac{1}{1-n}} (1-z/l_0)^{\frac{n+1}{n-1}} & \text{for} \quad z < l_0; \\ 0 & \text{for} \quad z \geqslant l_0. \end{cases} \tag{9}$$

Here $P \geqslant 0$ is an arbitrary constant determining the specific conditions of motion of the plate, the velocity of which decreases with time; the constant A is given by (6). Function (9) is the solution of problem (1) for an arbitrary value of the support dimension l_0.

THEOREM 3. Let $l_0 = z_+(0)$ be the dimension of the support of the initial distribution in problem (1) and let $B>0$ be an arbitrary constant. Then, if

$1^0.$ $u_0(z) \leqslant B \left(1 - \frac{z}{l_0}\right)^{\frac{n+1}{n-1}}$ for $z \leqslant l_0$;

$2^0.$ $u_1(t) \leqslant B \left(1 - \frac{t}{T_0}\right)^{\frac{1}{1-n}}$ for $t < T_0 = B^{1-n}\left[\frac{l_0^{n+1}}{2n}\left(\frac{n-1}{n+1}\right)^n\right]$,

the solution of problem (1) — $u(t, z)$ — is localized at least during an interval of time $\Delta t \geqslant T_0$.

Proof. Consider the function

$$\tilde{w}(t, z) = \begin{cases} B\left(1 - \frac{t}{T_0}\right)^{\frac{1}{1-n}} \left(1 - \frac{z}{l_0}\right)^{\frac{n+1}{n-1}} & \text{for} \quad z < l_0, \\ 0 & \text{for} \quad z \geqslant l_0, \end{cases} \tag{10}$$

which is the solution in the region $S=R_+^2\cap\{0\leqslant t<T_0\}$ of the boundary-value problem

$$\tilde{w}_t=(|\tilde{w}_z|^{n-1}\tilde{w}_z)_z,$$
$$\tilde{w}(0,z)=B\left(1-\frac{z}{l_0}\right)^{\frac{n+1}{n-1}},\quad \tilde{w}(t,0)=B\left(1-\frac{t}{T_0}\right)^{\frac{1}{1-n}},\quad \tilde{w}(t,\infty)=0. \tag{11}$$

The monotonic dependence of the solution of problem (11) on the initial and boundary conditions implies that everywhere in S, $\tilde{w}\geqslant\tilde{u}$, where $\tilde{u}$ is the solution in S of the boundary-value problem

$$\tilde{u}_t=(|\tilde{u}_z|^{n-1}\tilde{u}_z)_z,\quad \tilde{u}(0,z)=u_0(z),\quad \tilde{u}(t,0)=u_1(t),\quad \tilde{u}(t,\infty)=0, \tag{12}$$

provided the initial and boundary conditions of problem (12) coincide with the corresponding conditions for u(t, z) and thus satisfy requirements 1° and 2° of the theorem.

However, Theorem 1 of [3] implies that $\tilde{u}\geqslant u$ everywhere in S. Thus, we have finally that $\tilde{w}\geqslant u$ everywhere in S, so that $\operatorname{supp} u\subset[0,l_0]$, for all $t\in[0,T_0)$. However, since $z_+(0)=l_0$ and $\dot{z}_+(t)\geqslant 0$, and by virtue of Theorem 1, we have $z_+(t)\equiv l_0$ and $\dot{z}_+(t)=0$ for all $t\in[0,T_0)$. The theorem is proved.

THEOREM 4. Suppose in problem (1) that

$$1^0\ u_0(z)\geqslant C(1-z/l_0)^{n/(n-1)}\quad \text{for}\quad z\leqslant l_0,$$
$$2^0\ u_1(t)\geqslant C\quad \text{for all}\ t\in[0,\varepsilon/2],$$

where $l_0=z_+(0)$ is the dimension of the support of the initial distribution $u_0(z)$, ε is a quantity satisfying the inequality $\varepsilon\leqslant n\gamma^{-1}(n^2-1)^{-1}$, and

$$C=\left\{\frac{2}{\varepsilon(n+1)}\left[\frac{l_0(n-1)}{n}\right]^{n+1}\right\}^{1/(n-1)}.$$

Then the solution of problem (1) cannot be metastably localized.

Proof. Consider in the half-strip $S_\varepsilon=R_+^2\cap\{0\leqslant t\leqslant\varepsilon/2\}$ the function

$$\tilde{v}(t,z)=\begin{cases} C\left[1-\dfrac{z}{l_0(1+2t/\varepsilon)^{\frac{1}{n+1}}}\right]^{\frac{n}{n-1}} & \text{for } z<l_0\left(1+\dfrac{2t}{\varepsilon}\right)^{\frac{1}{n+1}},\\ 0 & \text{for } z\geqslant l_0\left(1+\dfrac{2t}{\varepsilon}\right)^{\frac{1}{n+1}}.\end{cases}$$

As follows from [5], this function, in the region S_ε, is the solution of the boundary-value problem

$$\tilde{v}_t=(|\tilde{v}_z|^{n-1}\tilde{v}_z)_z-\frac{n\tilde{v}}{(n^2-1)(t+\varepsilon/2)},$$
$$\tilde{v}(0,z)=C\left(1-\frac{z}{l_0}\right)^{\frac{n}{n-1}},\quad \tilde{v}(t,0)=C,\quad \tilde{v}(t,\infty)=0.$$

It then follows from the comparison theorem of [3] that $\tilde{v}(t,z)\leqslant u(t,z)$ everywhere in S_ε. Accordingly, $z_+(t)\geqslant l_0(1+2t/\varepsilon)^{1/(n+1)}$ for $\forall t\in[0,\varepsilon/2]$, and since $z_+(0)=l_0$, it follows that $\dot{z}_+(0)>0$. This inequality contradicts the definition of metastable localization. The theorem is proved.

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