EFFECT OF FINITE PERTURBATIONS ON STABILITY OF HARTMANN FLOW OF VISCOPLASTIC FLUID

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The stability of a flow of a conducting viscous plastic under small disturbances in a perpendicular magnetic field is investigated in [1]. It was shown that Hartmann flow of a viscous plastic is stable under disturbances of infinitely small amplitude of $Ha<Ha_* = 6.5$, while stability is lost if $Ha>Ha_*$.

In the present paper we consider the stability of a flow of a conducting viscoplastic fluid in a perpendicular magnetic field under disturbances of finite amplitude.

An unsteady flow of a viscoplastic fluid in a magnetic field is described by the equations of magnetohydrodynamics [2]:

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{u}\nabla\right)\mathbf{u}=-\nabla p+\nabla\sigma_{ij}+\frac{1}{\mu_0}[[\nabla\times\mathbf{B}]\times\mathbf{B}];$$
$$\frac{\partial\mathbf{B}}{\partial t}=[\nabla\times[\mathbf{u}\times\mathbf{B}]]+\frac{1}{\mu_0\sigma}\triangle\mathbf{B}, \tag{1}$$

where σ is the conductivity of the medium, ρ is the density, and μ_0 is the magnetic permeability. The deviator of the stress tensor σ_{ij} is defined in accordance with the rheological equation of a viscoplastic medium [3]:

$$\sigma_{ij}=2\left(\mu+\frac{\tau_0}{\sqrt{2f_{lm}f_{lm}}}\right)f_{ij}\quad\text{for}\quad\sqrt{\sigma_{lm}\sigma_{lm}}\geqslant\tau_0;$$
$$f_{ij}=0\quad\text{for}\quad\sqrt{\sigma_{lm}\sigma_{lm}}<\tau_0, \tag{2}$$

where $f_{ij} = \frac{1}{2}[(\partial u_i/\partial x_j) + (\partial u_j/\partial x_i)]$ is the strain-rate tensor, μ is the coefficient of dynamic viscosity, and τ_0 is the yield point (the limiting case $\tau_0 \to 0$ corresponds to the case of a Newtonian fluid).

We introduce the following as characteristic quantities: the half-width of the channel L; the velocity $u^0 = [-\partial p/\partial x + \sigma E_0B_0]L^2/2\mu$; and the external magnetic field B_0. For a steady isotropic flow in a planar channel ($-1 \leqslant y \leqslant 1$) in a transverse magnetic field, we then obtain the following symmetric distribution of the dimensionless velocity [4]:

$$u_0(y)=\begin{cases}\dfrac{2}{\mathrm{Ha}^2}\left[1-\dfrac{\mathrm{ch\,Ha}(\xi+y)}{\mathrm{ch\,Ha}(1-\xi)}\right] & \text{for}\quad -1\leqslant y\leqslant-\xi;\\[2ex] \dfrac{2}{\mathrm{Ha}^2}\left[1-\dfrac{1}{\mathrm{ch\,Ha}(1-\xi)}\right] & \text{for}\quad -\xi\leqslant y\leqslant 0,\end{cases} \tag{3}$$

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where ξ is the dimensionless half-width of the quasisolid zone, determined from the transcendental equation

$$\mathrm{ch}\,\mathrm{Ha}\,(1-\xi)=2\xi/\varkappa. \tag{4}$$

Here $\mathrm{Ha}=B_0L\sqrt{\sigma/\mu}$ is the Hartmann number and $\varkappa = \tau_0 L/\mu_0 U^0$ is a dimensionless parameter characterizing the plastic properties of the viscous plastic.

We assume that the following dimensionless perturbations of the velocity and magnetic field are imposed on flow (3):

$$\begin{aligned}\varphi(x,y,t)&=\varphi_0(y)+{}^1/_2a[\varphi(y)e^{i\alpha(x-ct)}+\tilde{\varphi}(y)e^{-i\alpha(x-ct)}];\\ \psi(x,y,t)&=\psi_0(y)+{}^1/_2a[\psi(y)e^{i\alpha(x-ct)}+\tilde{\psi}(y)e^{-i\alpha(x-ct)}].\end{aligned} \tag{5}$$

Here α and αc are the wave number and complex frequency of the disturbances; a is the amplitude of the nonstationary part of the disturbances; The functions $\varphi_0(y)$ and $\psi_0(y)$ define, respectively, the stationary distortion of the velocity profile of main flow (3) and of the induced magnetic field caused by the finite disturbance, while the functions $\varphi(y)$ and $\psi(y)$ denote, respectively, the distributions of the nonstationary parts of the perturbations of velocity and magnetic field across the channel; the sign $\sim$ denotes the complex conjugate.

We assume that the pressure gradient averaged over a period $\langle\partial p/\partial x\rangle$ equals the pressure gradient of the undisturbed flow. Then, inserting (5) into the basic system of equations of magnetic rheodynamics (1), we obtain, accurate to terms of order a^2 but allowing for terms of order $a^2\mathrm{Re}$, the following system of equations for $\varphi_0(y)$, $\psi_0(y)$, $\varphi(y)$, and $\psi(y)$:

$$\begin{gathered}\mathfrak{L}\varphi+\frac{4\alpha^2\varkappa}{i\alpha\,\mathrm{Re}}D[D\varphi(Du)^{-1}]=\frac{i\,\mathrm{Ha}^2}{\alpha\,\mathrm{Re}}D^2\varphi;\\ \mathrm{Re}^{-1}D^3\varphi_0(y)+\mathrm{Al}\,D^2\psi_0(y)={}^1/_4i\alpha a^2D[(\tilde{\varphi}D\varphi-\varphi D\tilde{\varphi})-\mathrm{Al}(\tilde{\psi}D\psi-\tilde{\psi}D\psi)];\\ i(\alpha\,\mathrm{Re}_m)^{-1}[(D^2-\alpha^2)-(u-c)]\psi=(D+i\alpha B)\varphi;\\ \mathrm{Re}_m^{-1}D^3\psi_0(y)-D^2\varphi_0(y)={}^1/_4i\alpha a^2D^2(\tilde{\varphi}\psi-\varphi\tilde{\psi});\\ u=u_0(y)+D\varphi_0(y);\quad B=B(y)+D\psi_0(y);\end{gathered} \tag{6}$$

here

$$\mathfrak{L}\equiv(u-c)(D^2-\alpha^2)-D^2u+i(\alpha\,\mathrm{Re})^{-1}(D^2-\alpha^2)$$

is the Orr–Sommerfeld operator; $\mathrm{Re} = \rho u^0 L/\mu$ is the Reynolds number; $\mathrm{Al} = B_0^2/\mu\rho u^{02}$ is the Alfvén number; $\mathrm{Re_m} = \mu_0\sigma u^0 L$ is the magnetic Reynolds number; and $D\equiv d/dy$. If $\mathrm{Re_m}\ll 1$, magnetic field perturbations can be neglected compared with velocity perturbations, and system (6) acquires the form

$$\mathfrak{L}\varphi+\frac{4\alpha^2\varkappa}{i\alpha\,\mathrm{Re}}D[D\varphi(Du)^{-1}]=\frac{i\,\mathrm{Ha}}{\alpha\,\mathrm{Re}}D^2\varphi; \tag{7}$$

$${}^1/_4i\alpha a^2\,\mathrm{Re}\,D(\tilde{\varphi}D\varphi-\varphi D\tilde{\varphi})=D^3\varphi_0(y). \tag{8}$$

The functions $\varphi_0(y)$ and $\varphi(y)$ must satisfy the "no-slip" conditions on the solid surface of the channel ($y = -1$):

$$D\varphi_0(-1)=0;\quad \varphi(-1)=D\varphi(-1)=0. \qquad (9),\ (10)$$

Disturbance of main flow (3) leads to a deformation of the surface separating the zones of quasisolid and viscous flows. For a mean pressure gradient over the length of the channel ($\langle dp/dx\rangle$ = const) equal to the pressure gradient of the undisturbed flow, the expression for deformation of the surface separating the zones has the form

$$y=-\xi+\frac{a}{2}[he^{i\alpha(x-ct)}+\tilde{h}e^{-i\alpha(x-ct)}], \tag{11}$$

where $h = -\varphi''(-\xi)\ [u''(-\xi)]^{-1}$ is the deformation of the separating surface [5].

Perturbations of velocity and the strain-rate tensor f_{ij} reduce to zero on the boundary of the quasisolid zone $f_{ij} = 0$. These conditions, written on the mean surface separating the zones $y = -\xi$, have the following form [5], accurate to terms of order α^2:

$$\varphi_0(-\xi)=0; \quad \varphi(-\xi)=\varphi'(-\xi)=0. \qquad (12);\ (13)$$

We introduce a new independent variable $z = (y + \xi)/(1 - \xi)$, and at the same time we change the characteristic length L and velocity u° to $L_1 = L(1 - \xi)$ and $u_1 = U^\circ\{(2/Ha^2)[1 - (1/\cosh Ha(1 - \xi))]\}$. The problem is then reduced to investigating the stability of a Hartmann flow under finite disturbances [6]:

$$u_0(y)=\frac{\operatorname{ch} Ha_1-\operatorname{ch} Ha_1 z}{1-\operatorname{ch} Ha_1}; \quad Ha_1=B_0L_1\sqrt{\frac{\sigma}{\mu}}. \qquad (14)$$

Let us consider the solution of system (7), (8) subject to boundary conditions (9)-(13). The function $\varphi(z)$ is determined from the problem (7), (10), (13). The first pair of independent particular solutions φ_1 and φ_2 can be represented in the form of power series [5]:

$$\varphi_1=(z-z_c)\sum_{k=1}^{\infty}a_k(z-z_c)^{k-1}; \quad \varphi_2=P_c\ln(z-z_c)+\sum_{k=0}^{\infty}b_k(z-z_c)^k. \qquad (15)$$

Here z_c is the point at which $u(z_c) = \Re c$, $P_c = D^2u(z_c)/Du(z_c) - 2\alpha_2 = 2b$; z_c will be regarded as an independent parameter defining the coefficients in expansions (15). The second pair of independent particular solutions $\varphi_{3,4}$ (viscous integrals [7]) we take in the form

$$\varphi_{3,4}=\int_{\pm\infty}^{\eta}d\eta\int_{\pm\infty}^{\eta}\sqrt{\eta}H_{1/3}^{(1,2)}\left[\frac{2}{3}(i\eta)^{3/2}\right]d\eta;$$
$$\eta=(z-z_c)\varepsilon, \quad \varepsilon=(\alpha \operatorname{Re} u_c')^{1/3}. \qquad (16)$$

When constructing the solutions of Eq. (7), it must be remembered that the point $z = 0$ is a singular point of Eq. (7), since $u'(0) = 0$. Accordingly, in the neighborhood of the point $z = -o + O(\gamma)$, $\gamma = \varkappa(\alpha Re)^{-1/2}$, where the plastic properties manifest themselves, we transform Eq. (7) by means of a new independent variable $t = z\gamma^{-1}$ to the form [1]

$$\varphi^{IV}+\frac{A_1}{t}\varphi''+\frac{A_2}{t^2}\varphi'+A_3\varphi=0. \qquad (17)$$

Here the A_i are regular functions of t.

For $t \in [0, t_s]$, $t_s = z_s\gamma^{-1} \sim O(1)$, the quantity $A_3 \approx 0$ and the solutions of (17) have the form

$$\varphi_1^*=\text{const}; \quad \varphi_2^*=\sum_{k=0}^{\infty}\frac{\alpha_k t^{k+3}}{k+3}; \quad \varphi_3^*=\sum_{k=0}^{\infty}\frac{\beta_k t^{k+2}}{k+2};$$
$$\varphi_4^*=\int\varphi_3^*\ln t\,dt+\sum_{k=0}^{\infty}\frac{\gamma_k t^{k+1}}{k+1}, \qquad (18)$$

where α_k, β_k and γ_k are given by recurrence relations [1]. It follows from boundary conditions (13) that for $t \in (0, t_s)$ the general solution of Eq. (17) has the form

$$\varphi^*(t)=C_2^*\varphi_2^*+C_3^*\varphi_3^*. \qquad (19)$$

The condition that the general solution of Eq. (7) in the region [0, −1] be nontrivial, written allowing for boundary conditions (10) and (13) and for the requirement that the solutions φ and φ^* match at the point $z = z_s$,

$$\frac{d^k\varphi}{dy^k}(z_s)=\frac{d^k\varphi^*}{dy^k}(z_s) \qquad (k=0, 1, 2, 3), \qquad (20)$$

leads, accurate to terms of order $\varkappa(\alpha\mathrm{Re})^{-1/2}$, to the following secular equation:

$$\frac{\varphi_1(-1)\varphi_2(0)-\varphi_1(0)\varphi_2(-1)}{\varphi_2(0)\varphi_1'(-1)-\varphi_1(0)\varphi_2'(-1)}=\frac{\varphi_3(-1)}{\varphi_3'(-1)}, \tag{21}$$

which coincides in form which the secular equation in the study of the stability of a Hartmann flow with respect to antisymmetric perturbations [1].

Utilizing Eq. (8), boundary conditions (9) and (12), and the definitions of P_c and ε, we obtain the following relationships for the disturbance amplitude and the corresponding Reynolds number:

$$a^2=\frac{(1+z_c)^3}{\eta^3}\,\frac{(u_c'P_c-u_c'')}{\tau_c'}\,u_c'; \tag{22}$$

$$\mathrm{Re}=\frac{\eta^3}{\alpha(1+z_c)^3}\,\frac{(\tau_c'-P_c\tau_c)}{(u_c'\tau_c'-u_c''\tau_c)}, \tag{23}$$

where $\tau_c={}^1\!/_4 i[\varphi'(z_c)\tilde{\varphi}(z_c)-\tilde{\varphi}'(z_c)\varphi(z_c)]$.

Here $\mathrm{Re}=\rho u_1 L_1/\mu$. If we take as the characteristic velocity of motion of the quasisolid zone u_1 and as the characteristic dimension the half-width of the channel L, then the corresponding Reynolds number $\mathrm{Re}_*=\rho u_1 L/\mu$ and the disturbance amplitude a_*^2 are given by the expressions

$$\mathrm{Re}_*=\mathrm{Re}(1-\xi)^{-1};\quad a_*^2=a^2(1-\xi)^{-2}. \tag{24}$$

Utilizing secular equation (21) for fixed values of the Hartmann number Ha_1 and the parameter P_c to determine the values of α, z_c, and ε corresponding to neutral disturbances $\Im m\, c=0$, and inserting these values into (22) and (23), we find the function $\mathrm{Re}=\mathrm{Re}(\alpha, \mathrm{Ha}_1=\mathrm{const})$ defining the curve of neutral stability. The results of the calculations are shown in Fig. 1, where curves 1-4 correspond to $\mathrm{Ha}_1=0$, $P_c=-2.12$, -2.1, -2.08, and -2.6; curves 5-8 to $\mathrm{Ha}_1=2$, $P_c=-2.66$, -2.64, -2.62, and -2.6; curves 9-11 to $\mathrm{Ha}_1=3$, $P_c=-3.4$, -3.38, and -3.36; curves 12-15 to $\mathrm{Ha}_1=4$, $P_c=-5.62$, -5.6, -5.48, and -5.46.

Plots of the critical values of the Reynolds number Re_0, the square of the disturbance amplitude a_0^2, and the wave number α_0 versus the Hartmann number Ha_1 are shown in Fig. 2. The convergence of series (15) deteriorates greatly at large values of the Hartmann number ($\mathrm{Ha}_1>5$). In this case the velocity distribution in the flow becomes similar to the velocity distribution in a boundary layer of thickness Ha_1^{-1} with an almost uniform velocity in the center of the channel; it is given by

$$\bar{u}(z)=1-e^{-\delta}, \tag{25}$$

where $\delta=\mathrm{Ha}_1(1+z)$ for $-1\leqslant z\leqslant 0$. In this case Eq. (7) can be rewritten in the form

$$\mathfrak{L}\varphi+\frac{4\alpha^*\varkappa}{i\,\mathrm{Re}^*}D[D\varphi(D\bar{u})^{-1}]=\frac{i}{\alpha^*\,\mathrm{Re}^*}D^2\varphi, \tag{26}$$

where

$$\alpha^*=\alpha\,\mathrm{Ha}_1^{-1},\quad \mathrm{Re}=\mathrm{Re}^*\,\mathrm{Ha}_1^{-1},\quad D\equiv d/d\delta.$$

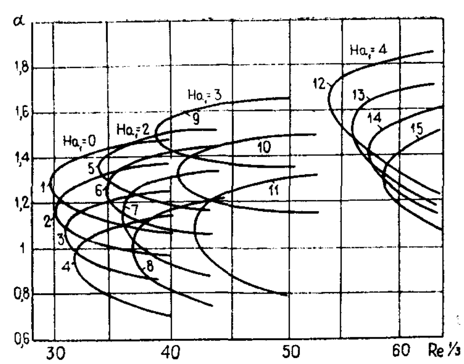

Fig. 1. Curves of neutral stability for $\mathrm{Ha}_1=0$, 2, 3, and 4.

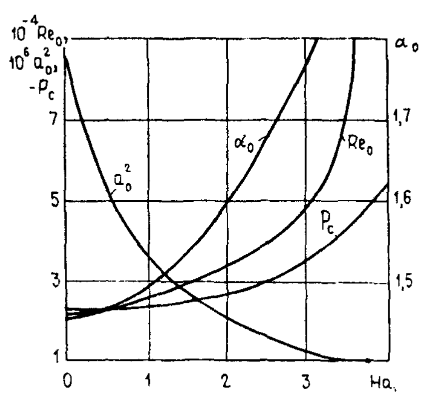

Fig. 2

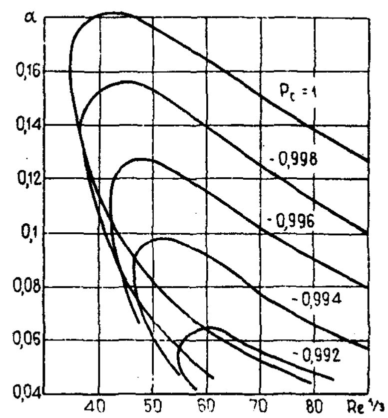

Fig. 3

Fig. 2. Dependence of critical Reynolds number Re_0 and corresponding values of square of disturbance amplitude a_0^2, wave number α_0, and parameter P_c on Hartmann number Ha_1.

Fig. 3. Curves of neutral stability for $Ha_1 \gg 1$.

The solutions of Eq. (26) are identical to those of Eq. (7), except that z, α, and Re are replaced by δ, α*, and Re*. Under this substitution, z = 0 goes over into $\delta = Ha_1$ and for $\delta = Ha_1 = 5$ can be regarded as the edge of the boundary layer, since for $\delta \geqslant 5$ the velocity profile can be regarded with sufficient accuracy as equal to unity. On the edge of the boundary layer, we can specify the following boundary conditions:

$$\varphi' + \alpha^*\varphi = 0 \quad \text{for} \quad \delta = 5. \tag{27}$$

On the wall δ = 0 (z = −1) we impose the standard "no-slip" conditions:

$$\varphi = D\varphi = 0. \tag{28}$$

In this case the secular equation has the following form, accurate to $(\alpha Re)^{-1}$:

$$\frac{\varphi_2(0)[\varphi_1'(5)+\alpha^*\varphi_1(5)]-\varphi_1(0)[\varphi_2'(5)+\alpha^*\varphi_2(5)]}{\varphi_2'(0)[\varphi_1'(5)+\alpha^*\varphi_1(5)]-\varphi_1'(0)[\varphi_2'(5)+\alpha^*\varphi_2(5)]}=\frac{\varphi_3(0)}{\varphi_3'(0)}. \tag{29}$$

Utilizing Eq. (8), boundary conditions (27) and (28), and also the definitions of P_c and ε, we obtain the dependences for Re and a^2.

Calculations showed that for $Ha_1 \gg 1$ the magnetic field freezes finite disturbances, and the disturbance most dangerous as far as loss of stability is concerned are small disturbances corresponding to the linear theory. In Fig. 3 we show the curves of neutral stability for $Ha \gg 1$ and P_c = −1, −0.998, −0.996, −0.994, and −0.992. It can be seen that the minimum Reynolds number corresponds to an undistorted velocity profile (P_c = −1) and is given, along with the corresponding value of the wave number, by

$$Re = 5\cdot 10^4\, Ha_1 (1-\xi)^{-1}; \quad \alpha = 0.19\, Ha_1 (1-\xi)^{-1}. \qquad (30),\ (31)$$

The effect of finite disturbances in a strong magnetic field leads to the appearance of subcritical instability. Evidently, this is connected with the ability of a magnetic field to suppress turbulent exchange of energy between disturbances of different scales.

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