SIMULTANEOUS CALCULATION OF THE TRANSVERSE AND LONGITUDINAL FRINGE EFFECTS IN THE CHANNEL OF A PLANE INDUCTION-TYPE MHD PUMP

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Introduction. In design calculations for plane induction-type MHD pumps under the electrodynamic approximation [1, 2], the liquid metal is represented by a moving, electrically conducting strip of finite width but infinite length, while the magnetic conductor is envisioned as having an active (coil wound) and a shunted (not wound) zone.

This approach makes it possible to take into account both the effect arising from the finite channel width (transverse fringe effect) and that from the finite magnetic conductor length (longitudinal fringe effect).

In the hydrodynamic analysis [3, 4], as a rule, one uses a model wherein the pump under consideration is replaced by a section of a pump having an infinitely long inductor and where the flowing medium in the pump is represented by a large number of narrow strips of liquid metal moving at different velocities. In addition, the magnetic field is assumed to be moving in a simple manner and the supposition is made that there is no longitudinal fringe effect.

In the present work the influence of the transverse and longitudinal effects is taken into account in a model unifying both approaches. In this model the flowing part consists of moving, narrow, infinitely long strips of liquid metal, while the magnetic conductor has a finite length and consists of an active and a shunted section. The computational model corresponds to a pump construction with an arrangement of thin longitudinal, electrically conducting partitions in the channel.

The influence of the longitudinal and transverse effects on the flow in a different arrangement has already been considered in [5, 6].

1. Problem Statement, and Governing Equations. We consider, under the noninductive approximation, the plane parallel laminar flow of an electrically conducting liquid through *a channel of a plane induction pump without short-circuited side busbars, having insulated partitions* at the entry and at the exit. Because of the assumptions made, the analytical solution derived below is qualitative in nature. We do succeed, however, in evaluating the forces dependent on the longitudinal fringe effect, which arise from the interaction of the moving magnetic field with currents induced in the liquid metal by the pulsating component of the external magnetic field. The computational scheme is illustrated in Fig. 1. The current load is uniformly distributed in region I. There are no windings in regions II and III, and insulated partitions (zero conductivity zone, $\sigma = 0$) are located inside the channel. The vectors have the components:

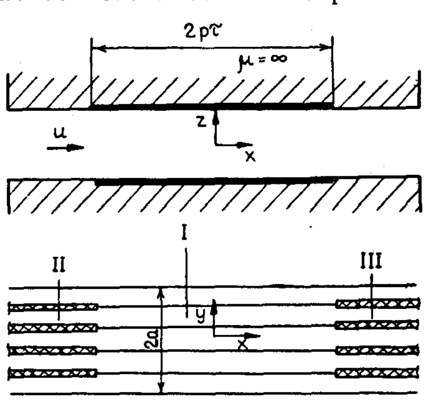

Fig. 1. Formulation of the problem.

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$$\mathbf{B}_0(0, 0, B_0(x)); \quad \mathbf{v}(u(y), 0, 0);$$

$$\mathbf{E}(E_x(x, y), E_y(x, y), 0); \quad \mathbf{j}(j_x(x, y), j_y(x, y), 0);$$

where the index "0" refers to the external field. The time dependence of all parameters is assumed to have the form:

$$A = \dot{A}e^{i\omega t}, \tag{1}$$

where $\dot{A}$ is the complex amplitude, which is a function of coordinates.

For a magnetic conductor having infinite magnetic permeability the following relationship holds [7]:

$$\dot{H}_0 = H_0(e^{-ix\alpha} + k_{III}^p), \tag{2}$$

where $k_{III}^p = (-1)^{p+1}k_{III}$, $k_{III} = (l - 2p\tau)/l$, l is the total inductor length, p is the number of pole pairs, τ is the polar division, $\alpha = \pi/\tau$, and H_0 is the amplitude of the moving component of the external field. For the noninductive approximation, one has:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}_0); \tag{3}$$

$$\text{curl } \mathbf{E} = -\partial \mathbf{B}_0/\partial t. \tag{4}$$

In addition, the equation of continuity for the current density also must be satisfied:

$$\text{div } \mathbf{j} = 0. \tag{5}$$

Equation (4) is satisfied if one sets:

$$\mathbf{E} = \mathbf{E}_1 + \nabla\varphi, \tag{6}$$

where the vector $\mathbf{E}_1$ has a unique component along the y axis, E_1, which equals:

$$E_1 = \frac{\mu\omega H_0}{\alpha}(e^{-i\alpha x} - i\alpha x k_{III}^p). \tag{7}$$

Using Eqs. (5), (3), (6), (7), and (2) we obtain the equation for the complex potential:

$$\frac{\partial^2 \dot{\bar{\varphi}}}{\partial \bar{x}^2} + L^2 \frac{\partial^2 \dot{\bar{\varphi}}}{\partial \bar{y}^2} = \dot{\bar{H}}_0 \frac{d\bar{u}}{d\bar{y}}. \tag{8}$$

Dimensionless quantities, identified by dashes, are introduced using the relationships:

$$\bar{y} = y\pi/a, \quad \bar{x} = x\alpha,$$

$$\dot{\bar{H}}_0 = \frac{\dot{H}_0}{H_0}, \quad \bar{u} = \frac{u\alpha}{\omega}, \quad \dot{\bar{\varphi}} = \frac{\alpha^3 a}{\mu H_0 \omega\pi}\varphi, \quad L = \frac{\tau}{a}.$$

After the introduction of $\bar{\varphi}_a$ and $\bar{\varphi}_r$ for the real and imaginary components of the complex potential, and leaving out the dashes above dimensionless quantities, we get:

$$\frac{\partial^2 \varphi_a}{\partial x^2} + L^2 \frac{\partial^2 \varphi_a}{\partial y^2} = (\cos x + k_{III}^p) W;$$
$$\frac{\partial^2 \varphi_r}{\partial x^2} + L^2 \frac{\partial^2 \varphi_r}{\partial y^2} = -\sin x \cdot W; \quad W(y) = \frac{du}{dy}. \tag{9}$$

We now formulate the boundary conditions for Eq. (9). On the insulated walls we set

$$j_y = 0, \tag{10}$$

from symmetry conditions it follows that at the channel center the following should hold:

$$j_x = 0, \tag{11}$$

while in regions II and III, on account of the assumption of no conductivity at $x = \pm p\tau$ we expect:

$$j_x = 0. \tag{12}$$

Employing Eqs. (10) to (12), (3) and also the condition that the velocity be zero on the wall and that there exists arbitrariness in the potential definition, we arrive at the boundary conditions for Eqs. (9) in the form:

$$x=\pm p\pi:\quad \frac{\partial\varphi_a}{\partial x}=\frac{\partial\varphi_r}{\partial x}=0;$$
$$y=\pi:\quad \frac{\partial\varphi_a}{\partial y}=-\frac{\cos x}{L^2};\quad \frac{\partial\varphi_r}{\partial y}=\frac{(\sin x+x)k_{\text{m}}{}^{\text{p}}}{L^2};\tag{13}$$
$$y=0:\quad \varphi_a=\varphi_r=0.$$

The value of the electrovolumetric force, averaged over a period, is computed, as usual, from the formula:

$$\langle f_e\rangle=0.5\,\Re\,[j_y\dot{B}_0{}^*].\tag{14}$$

After simple transformations, emitting the dashes above dimensionless quantities, we obtain:

$$\langle f_e\rangle=f_0\left\{(1+k_{\text{m}}{}^{\text{p}}\cos x+k_{\text{m}}{}^{\text{p}}x\sin x)+L^2\left[(\cos x+k_{\text{m}}{}^{\text{p}})\frac{\partial\varphi_a}{\partial y}-\sin x\frac{\partial\varphi_r}{\partial y}\right]-(1+2k_{\text{m}}{}^{\text{p}}\cos x+(k_{\text{m}}{}^{\text{p}})^2)u\right\},\tag{15}$$

where $f_0 = \sigma B_0^2 v_s/2$.

Averaging the $\langle f_e\rangle$ value over the channel length yields:

$$\langle f_e\rangle_{av}=\frac{1}{2p\pi}\int_{-p\pi}^{+p\pi}\langle f_e\rangle\,dx=f_0\left\{1+k_{\text{m}}+\frac{L^2}{2p\pi}\int_{-p\pi}^{+p\pi}\left[(\cos x+k_{\text{m}}{}^{\text{p}})\frac{\partial\varphi_a}{\partial y}-\sin x\frac{\partial\varphi_r}{\partial y}\right]dx-(1+k_{\text{m}}{}^2)u\right\}.\tag{16}$$

We now turn our attention to the equations of motion. The initial equation of motion for laminar flow, obtained by averaging over the channel height, has the form [3]:

$$\eta\frac{d^2u}{dy^2}-\frac{4\eta}{\delta^2}\frac{\text{Ha th Ha}}{1-\text{th Ha/Ha}}u+f_e-\frac{\partial p}{\partial x}=0.\tag{17}$$

After averaging Eq. (17) over time and channel length, and using Eq. (16), we obtain, in terms of dimensionless quantities:

$$\frac{d^2u}{dy^2}-qu+b\left\{1+k_{\text{m}}+\frac{L^2}{2\pi p}\int_{-p\pi}^{+p\pi}\left[(\cos x+k_{\text{m}}{}^{\text{p}})\frac{\partial\varphi_a}{\partial y}-\sin x\frac{\partial\varphi_r}{\partial y}\right]dx-(1+k_{\text{m}}{}^2)u\right\}-C=0.\tag{18}$$

Equation (18) can be interpreted as that governing the motion of a narrow strip of liquid metal. The q, b, C, and y_0 parameters appearing in it have the following meanings:

$$q=4\left(\frac{a}{\pi\delta}\right)^2\frac{\text{Ha}^2\,\text{th Ha}}{\text{Ha}-\text{th Ha}};\quad b=\frac{\sigma\mu_0{}^2H_0{}^2y_0{}^2}{2\eta};\quad C=\frac{y_0{}^2}{\eta v_s}\frac{\Delta p}{2p\pi};\quad y_0=\frac{a}{\pi};$$

and v_s is the velocity of the moving field.

The boundary conditions for the velocity u have the form:

$$u(\pi)=0;\quad du(0)/dy=0.\tag{19}$$

The problem thus has been reduced to the solution of the system of equations (9) and (18) with the boundary conditions (13) and (19).

2. Solution and Results. We shall seek a solution for velocity u(y) in the form of a Fourier series expansion in eigenfunctions of the boundary problem $Z'' = -\lambda Z$, $Z(\pi) = Z'(0) = 0$, i.e., in the form:

$$u(y)=\sum_{n=0}^{\infty}u_n\cos\xi_n y,\tag{20}$$

where $\xi = n + 0.5$.

The justification for such expansion follows from the Steklov theorem [8]. We solve Eq. (9) by the Grinberg method [9]. We write:

$$\varphi_r(x, y) = \sum_{n=0}^{\infty} \Phi_n{}^r(x) \cdot \sin \xi_n y; \quad \varphi_a(x, y) = \sum_{n=0}^{\infty} \Phi_n{}^a(x) \cdot \sin \xi_n y, \tag{21}$$

where

$$\Phi_n{}^r(x) = \frac{2}{\pi} \int_0^{\pi} \varphi_r(x, y) \cdot \sin \xi_n y dy; \quad \Phi_n{}^a(x) = \frac{2}{\pi} \int_0^{\pi} \varphi_a(x, y) \cdot \sin \xi_n y dy. \tag{22}$$

After multiplication of the equation for φ_r by $2\pi^{-1}\sin\xi_n y$ and integration from 0 to π, taking into account the boundary conditions (13), we obtain an equation for $\Phi_n{}^r(x)$ in the form:

$$\begin{gathered} \Phi_n{}^{r\prime\prime} - \beta_n{}^2 \Phi_n{}^r = -2\pi^{-1}(-1)^n (\sin x + x k_{\text{ш}} \text{p}) + \xi_n u_n \sin x; \\ \Phi_n{}^{r\prime}(\pm \text{p}\pi) = 0. \end{gathered} \tag{23}$$

In an analogous manner, for $\Phi_n{}^a(x)$ we have

$$\begin{gathered} \Phi_n{}^{a\prime\prime} - \beta_n{}^2 \Phi_n{}^a = \frac{2}{\pi}(-1)^n \cos x - \xi_n u_n (\cos x + k_{\text{ш}} \text{p}); \\ \Phi_n{}^{a\prime}(\pm \text{p}\pi) = 0 \qquad (\beta_n = L\xi_n). \end{gathered} \tag{24}$$

The solution can now be written in the form:

$$\Phi_n{}^a = -\frac{1}{1+\beta_n{}^2} \left\{ (-1)^n \frac{2}{\pi} - \xi_n u_n \right\} \cos x + \frac{k_{\text{ш}} \text{p}}{\beta_n{}^2} \xi_n u_n; \tag{25}$$

$$\begin{aligned} \Phi_n{}^r = & -\frac{\text{sh}\, \beta_n x}{\beta_n \,\text{ch}\, \beta_n \text{p}\pi} \left\{ \frac{2}{\pi} \frac{(-1)^n}{\beta_n{}^2} k_{\text{ш}} \text{p} + (-1)^{\text{p}} \left[\frac{2}{\pi}(-1)^n - \xi_n u_n \right] \times \right. \\ & \left. \times \frac{1}{1+\beta_n{}^2} \right\} + \left[\frac{2}{\pi}(-1)^n - \xi_n u_n \right] \frac{\sin x}{1+\beta_n{}^2} + \frac{2}{\pi} \frac{(-1)^n}{\beta_n{}^2} k_{\text{ш}} \text{p} x. \end{aligned} \tag{26}$$

The found expressions for φ_a and φ_r are now substituted into the equation of motion (18). At the same time, it is necessary to evaluate the values of the dimensionless electrovolumetric force:

$$\bar{f} = 1 + k_{\text{ш}} + \frac{L^2}{2\pi \text{p}} \int_{-\text{p}\pi}^{+\text{p}\pi} \left[(\cos x + k_{\text{ш}} \text{p}) \frac{\partial \varphi_a}{\partial y} - \sin x \frac{\partial \varphi_r}{\partial y} \right] dx - (1 + k_{\text{ш}}{}^2) u. \tag{27}$$

By using transformation identities and taking into account that:

$$\frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\cos \xi_n y}{\xi_n} = \begin{cases} 1, & 0 \leqslant y < \pi, \\ 0, & y = \pi, \end{cases} \tag{28}$$

$$\frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\cos \xi_n y}{\xi_n (1+\beta_n{}^2)} = 1 - \frac{\text{ch}\, y/L}{\text{ch}\, \pi/L} \qquad (0 \leqslant y \leqslant \pi), \tag{29}$$

it is possible to represent the electrovolumetric force in the form:

$$\bar{f} = \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[1 - \frac{\beta_n \,\text{th}\, \beta_n \text{p}\pi}{\text{p}\pi (1+\beta_n{}^2)} \right] \frac{(1 - {}^1\!/_2 \pi (-1)^n \xi_n u_n) \cos \xi_n y}{\xi_n (1+\beta_n)^2} + k_{\text{ш}} \frac{2L}{\text{p}\pi^2} \sum_{n=0}^{\infty} (-1)^n \frac{\text{th}\, \beta_n \text{p}\pi \cdot \cos \xi_n y}{\beta_n{}^2 (1+\beta_n{}^2)}. \tag{30}$$

After substitution of the electrovolumetric force values into Eq. (18), and taking into account Eqs. (20), (28), and (29), we find the u_n coefficients in the velocity equation (20):

$$u_n = \frac{(-1)^n \left[\frac{2bL}{\text{p}\pi^2} \frac{\text{th}\, \beta_n \text{p}\pi}{1+\beta_n{}^2} \left(\frac{1}{1+\beta_n{}^2} - \frac{k_{\text{ш}}}{\beta_n{}^2} \right) - \frac{2b}{\pi \xi_n (1+\beta_n{}^2)} + \frac{2C}{\pi \xi_n} \right]}{-\xi_n{}^2 - q - \frac{b}{1+\beta_n{}^2} + \frac{b\beta_n}{\pi \text{p}} \frac{\text{th}\, \beta_n \text{p}\pi}{(1+\beta_n{}^2)^2}}. \tag{31}$$

The (P, Q) characteristic can be obtained from the equation:

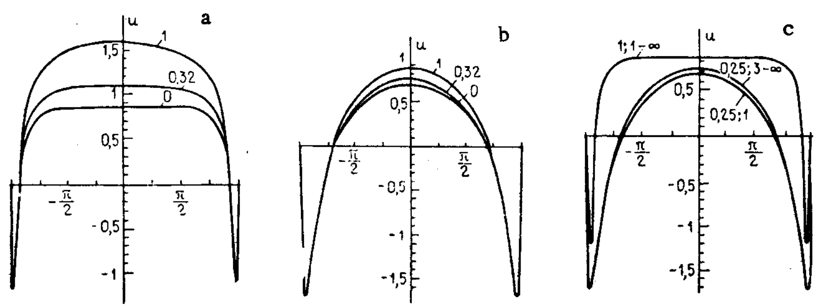

Fig. 2. Velocity profile distributions. The parameter in a) and b) is the value of the shunting coefficient; in c) the first parameter is the relative channel width a/τ, the second is the number of pole pairs.

$$u_{av}=\frac{1}{\pi}\int_0^{\pi} u\,dy=\frac{1}{\pi}\sum_{n=0}^{\infty} u_n\frac{(-1)^n}{\xi_n}, \tag{32}$$

where u_n's are dimensionless pressure gradient functions computed from Eq. (31).

The last term in Eq. (30), which comes about from the presence of the shunted region in the inductor, results in an increase in the electrovolumetric force. This result is in agreement with that obtained earlier under the electrodynamic approximation for small slip ([10] p. 37, Fig. 18).

In order to trace the effect of the end sections (the sections where the metal enters and leaves the active zone), we consider a particular case of Eq. (30).

1. The external field is purely of the moving kind (k_{III} = 0), and the influence of end sections is negligibly small (p = ∞), which corresponds to the customarily used model [3, 4]:

$$\bar{f}=\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{1}{\xi_n(1+\beta_n^2)}\left[1-\frac{\pi}{2}(-1)^n\xi_n u_n\right]\cos\xi_n y. \tag{33}$$

2. The number of pole pairs is finite, k_{III} = 0,

$$\bar{f}=\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[1-\frac{\beta_n \operatorname{th}\beta_n p\pi}{p\pi(1+\beta_n^2)}\right]\frac{(1-{}^1/_2\pi(-1)^n\xi_n u_n)\cos\xi_n y}{\xi_n(1+\beta_n^2)}. \tag{34}$$

It follows from a comparison of Eqs. (33) and (34) that the end sections always reduce the electrovolumetric force.

3. The infinitely wide channel (L → ∞):

$$\bar{f}=1+k_{III}-u. \tag{35}$$

As for p → ∞, in this case the end sections also do not influence the electrovolumetric force.

Equation (35), which illustrates the increase of the electrovolumetric force with rising k_{III}, canbe obtained directly from the electrodynamic approximation.

Shown in Figs 2a, b are calculated velocity profiles for various shunting coefficients and fixed pressure gradients, number of pole pairs, and the relative channel width a/τ. These include special velocity profiles with reversed flows at the insulated walls.

In the cases considered the longitudinal fringe effect is found to affect the velocity in the channel central portion and to have practically no effect on the reverse flows at the walls.

Figure 2c shows the effects of the number of pole pairs and of the relative channel width for k_{III} = 0. The influence of the end sections is small and becomes weaker as the pump gets wider. Thus, for a/τ = 0.25 the profiles run together starting with the number of pole pairs p = 3, while for $a/\tau = 1$ one sees a coincidence of profiles practically over the entire range of pole pair numbers.

CONCLUSIONS

1. A mathematical model has been presented which represents a plane induction type MHD pump and allows a simultaneous computation of the influence both of the longitudinal and transverse boundary effects on the flow in this channel.

2. An analytical solution has been found under the noninductive approximation for the steady laminar flow through a channel having insulating partitions at the entry and the exit.

3. In the examples treated the longitudinal boundary effect influences the velocity in the central channel portion and is found not to affect the reverse flow regions near the walls.

4. Under the given approximation, the inclusion of the pulsating magnetic field component results in an increased flow rate for a fixed pressure gradient. Compared to the case of an infinitely long pump, the entry and exit regions are found to decrease the flow rate. The influence of these regions gets smaller with increasing a/τ ratio.

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