CALCULATION OF TWO-DIMENSIONAL ELECTROMAGNETIC FIELDS IN CHANNELS OF INDUCTION MHD MACHINES WITH AN OPEN MAGNETIC DUCT FOR FINITE Re_m NUMBERS. II

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In the present part of our work the method of calculating the electromagnetic field in a planar MHD machine with an open magnetic duct is extended to axially symmetric machines. An integral equation is derived, describing the current density distribution in a cylindrical MHD channel, whose solution is constructed in the form of an expansion in eigenfunctions of the integral operator. The algorithm suggested makes it possible, when calculating the electromagnetic field, to account for the secondary longitudinal edge effect and for the inhomogeneity in velocity along the channel height.

1. Consider the motion of a conducting fluid in a coaxial channel, located at the opening of a cylindrical inductor of finite sizes (Fig. 1). It is assumed that the induction of the external travelling field in the gap (inductor field) and the velocity distribution in the channel are given, and are

$$\dot{\mathbf{B}}_0(r, z) = \mathbf{r}^0 \dot{B}_r(r, z) + \mathbf{z}^0 \dot{B}_z(r, z),$$

$$\mathbf{v}(r, z) = \mathbf{z}^0 v(r).$$

We also assume that the channel walls are nonconducting, and the fluid conductivity is γ = const.

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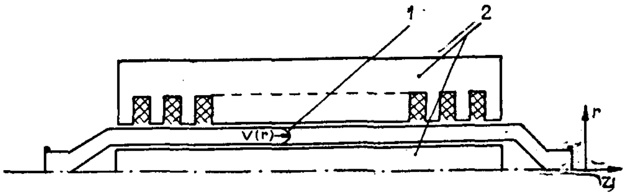

Fig. 1. Cross section of axially symmetric MHD channel by plane α = const: 1) velocity profile in the channel; 2) inductor with winding.

These assumptions imply that the electromagnetic field in the channel is axially symmetric, independently of the coordinate in a cylindrical coordinate system (r, α, z).

The equations describing the electromagnetic field in the channel are [1]

$$\operatorname{curl} \dot{\mathbf{B}} = \mu_0 \dot{\delta}; \quad \operatorname{curl} \dot{\mathbf{E}} = -j\omega \dot{\mathbf{B}}; \qquad \dot{\delta} = \gamma(\dot{\mathbf{E}}_s - j\omega \dot{\mathbf{A}} + [\mathbf{v} \operatorname{curl} \dot{\mathbf{A}}]; \tag{1}$$

here the vectors denoted by dots have complex values of the corresponding sinusoidal functions of time; $\{\operatorname{curl}\dot{\mathbf{A}} = \dot{\mathbf{B}};\ \operatorname{div}\dot{\mathbf{A}} = 0\}$; ω is the circular frequency, and $\dot{E}_s$ is the potential component of the electric field $\dot{\mathbf{E}}$.

Because of axial symmetry $\dot{E}_s = 0$. Indeed, $\oint_L \dot{\mathbf{E}}_s d\mathbf{l} = 0$, but in the present case $\dot{\mathbf{E}}_s = \alpha^0 \dot{E}_s$, $\dot{E}_s$ = const. Consequently, $\oint_L \dot{\mathbf{E}}_s d\mathbf{l} = \dot{E}_s \oint_L dl = 0$, whence $\dot{E}_s = 0$. We represent the magnetic field in the form $\dot{\mathbf{B}}(M) = \dot{\mathbf{B}}_0(M) + \dot{\mathbf{B}}_i(M)$, where $\dot{\mathbf{B}}_0$ and $\dot{\mathbf{B}}_i$ are the external and induced fields. From the first of Eqs. (1) it follows that [1]

$$\dot{\mathbf{A}}_i(M) = \frac{\mu_0}{4\pi} \int_{V_k} \frac{\dot{\delta}(N)}{r_{MN}} \, dV_N, \tag{2}$$

where V_k is the domain occupied by the conducting fluid, and $\dot{\mathbf{A}}_i$ is the vector potential of the field $\dot{\mathbf{B}}_i$. We note that the integral over the infinite volume in Eq. (2) exists, since $\dot{\delta}(N) \to 0$ for $N \to \infty$.

Consider equality (2) in the cylindrical coordinate system (r, α, z) whose z axis coincides with the longitudinal channel axis (Fig. 1). Rewriting the integral in (2) as a multiple one, and applying to both sides of the equality a Fourier transform in the coordinate z_M, we obtain

$$\dot{\mathbf{A}}_i(M) = \frac{\mu_0}{4\pi} \int_{S_k} \left[\int_{-\infty}^{\infty} \frac{\dot{\delta}(N)}{r_{MN}} \, dz_N \right] dS_N,$$

where S_k is the channel cross section by the plane α = const and

$$\int_{-\infty}^{\infty} \dot{\mathbf{A}}_i(r_M, \alpha_M, z_M) e^{jmz_M} dz_M = \frac{\mu_0}{4\pi} \int_{S_k} \left[\int_{-\infty}^{\infty} \dot{\delta}(N) \left(\int_{-\infty}^{\infty} \frac{e^{jmz_M} dz_M}{\sqrt{r_M^2 + r_N^2 - 2r_M r_N \cos(\alpha_M - \alpha_N) + (z_M - z_N)^2}} \right) dz_N \right] dS_N.$$

Performing the replacement $z_M - z_N = \xi$, $dz_M = d\xi$, $e^{jmz_M} = e^{jm\xi} \cdot e^{jmz_N}$, we then have

$$\begin{aligned} \dot{\mathbf{A}}_{im}(M) &= \frac{\mu_0}{4\pi} \int_{S_k} \dot{\delta}_m(N) \left(\int_{-\infty}^{\infty} \frac{e^{jm\xi} d\xi}{\sqrt{r_M^2 + r_N^2 - 2r_N r_M \cos(\alpha_M - \alpha_N) + \xi^2}} \right) dS_N = \\ &= \frac{\mu_0}{4\pi} \int_{S_k} \dot{\delta}_m(N) \cdot K_0(m r_{MN}) \, dS_N, \end{aligned} \tag{3}$$

where $\dot{\mathbf{A}}_{im}$, $\dot{\delta}_m$ are the Fourier transforms of the functions $\dot{\mathbf{A}}_i$, $\dot{\delta}$; $K_0(mr_{MN})$ is the MacDonald function, and

$$r_{MN} = \sqrt{r_M^2 + r_N^2 - 2r_M r_N \cos(\alpha_M - \alpha_N)}, \quad M, N \in S_k.$$

Multiplying Eq. (3) scalarly by $\alpha^0(M)$, and taking into account that $\dot{\mathbf{A}}_{im}$ and $\dot{\delta}_m$ have only an α-component independent of the coordinate α, we rewrite Eq. (3) in the form

$$\dot{A}_{im}(r_M)=\frac{\mu_0}{2\pi}\int\limits_{R_1}^{R_2}\dot{\delta}_m(r_N)r_N\mathfrak{K}(M,N)\,dr_N, \tag{4}$$

where

$$\mathfrak{K}(M,N)=\int\limits_0^{2\pi}\cos\alpha K_0(m\sqrt{r_M^2+r_N^2-2r_Nr_M\cos\alpha})\,d\alpha.$$

We apply a Fourier transform in the coordinate z_M to the projection of the third of Eqs. (1) on the α axis

$$\dot{\delta}_m(r)=-\gamma(j\omega\dot{A}_m(r)-jmv(r)\dot{A}_m(r))=-\gamma[j\omega\dot{A}_{im}(r)-jmv(r)\dot{A}_{im}(r)]+j\dot{f}_m, \tag{5}$$

where

$$\dot{f}_m=-\gamma(\omega+v(r))\dot{A}_{0m}.$$

Substitution of (4) into (5) leads to an integral equation of the second kind in $\dot{\delta}_m$:

$$\dot{\delta}_m(r_M)+\frac{\mu_0\gamma j}{2\pi}\int\limits_{R_1}^{R_2}\dot{\delta}_m(r_N)r_N[\omega-v(r_M)m]\mathfrak{K}(M,N)\,dr_N=j\dot{f}_m(r_M). \tag{6}$$

We rewrite this equation in relative units, while the notation for the relative units is retained:

$$\delta_m(r_M)+\frac{j\,\mathrm{Re}_m}{2\pi}\int\limits_{R_1}^{R_2}\dot{\delta}_m(r_N)r_N\varphi(r_M,m)\mathfrak{K}(M,N)\,dr_N=j\,\mathrm{Re}_m\dot{f}_m(r_M). \tag{7}$$

Here $\mathrm{Re}_m = (\mu_0\gamma s\omega\tau^2)/2\pi^2$; s is the slip, τ is polar fission, and

$$\varphi(r_M,m)=\omega-v(r_M)m.$$

We note that the right-hand side of (7) can be expressed in terms of the external field induction

$$\dot{f}_m(r_M)=\frac{\varphi(r_M,m)}{jm}\dot{B}_{0m}(r_M).$$

2. We seek a solution of Eq. (7) among the set of complex functions square-integrable in the interval $[R_1, R_2]$. On this function set we introduce a scalar product and a norm by the equalities

$$(\dot{\xi},\dot{\psi})=\int\limits_{R_1}^{R_2}\frac{r}{|\varphi(r,m)|}\dot{\xi}(r)\dot{\psi}(r)\,dr,\quad \|\xi\|^2=(\dot{\xi},\dot{\xi}). \tag{8}$$

The space of functions thus obtained is denoted by $L_{2\varphi}$. In this space the integral operator generated by Eq. (7) possesses the following properties:

1) in $L_{2\varphi}$ the integral operator $\mathfrak{K}$ is self-adjoint, i.e., $(\mathfrak{K}\dot{\xi},\dot{\psi})=(\dot{\xi},\mathfrak{K}\dot{\psi})$, where

$$\mathfrak{K}\dot{\xi}=\int\limits_{R_1}^{R_2}\xi(N)r_N\varphi(r_N,m)\mathfrak{K}(M,N)\,dr_N;$$

2) for m satisfying the inequality $\int\limits_{R_1}^{R_2}\frac{rf_k^2(r)}{\varphi(r,m)}dr>0$, the characteristic numbers of the operator $\mathfrak{K}$ are positive, while in the opposite case they are negative; here $f_k(r)$ is the k-th eigenfunction of the operator $\mathfrak{K}$;

3) the operator $\mathfrak{K}$ is completely continuous in $L_{2\varphi}$;

4) in the class $L_{2\varphi}$ the operator $\mathfrak{K}$ is complete.

The proof of these properties is similar to the proof given in the first part of this work.

These properties of the integral operator $\mathfrak{K}$ make it possible to represent the solution of Eq. (7) in form of an expansion in the eigenfunctions of the operator [2]. Putting

$$\dot{\delta}_m = \sum_{k=1}^{\infty} \dot{a}_k f_k; \quad \dot{f}_m = \sum_{k=1}^{\infty} \dot{b}_k f_k,$$

Eq. (7) is rewritten in the form

$$\sum_{k=1}^{\infty} \dot{a}_k f_k + \lambda \mathfrak{K} \sum_{k=1}^{\infty} \dot{a}_k f_k = j\,\mathrm{Re}_m \sum_{k=1}^{\infty} \dot{b}_k f_k, \quad \lambda = \frac{j\,\mathrm{Re}_m}{2\pi}; \quad \dot{b}_k = (\dot{f}_m, f_k). \tag{9}$$

Taking into account that $\mathfrak{K} f_k = f_k/\lambda_k$, we obtain an equation for finding $\dot{a}_k$:

$$\dot{a}_k = j \frac{\mathrm{Re}_m \dot{b}_k \lambda_k}{\lambda_k + \lambda}. \tag{10}$$

Consequently,

$$\dot{\delta}_m = j\,\mathrm{Re}_m \sum_{k=1}^{\infty} \frac{\lambda_k (\dot{f}_m, f_k) f_k}{\lambda_k + \lambda}. \tag{11}$$

From Eq. (11) follows the existence of solutions of Eq. (7) for arbitrary Re_m and f_m. Indeed, due to the properties of the operator $\mathfrak{K}$, λ_k are real numbers, and λ is a purely imaginary number. Therefore, the denominator in (11) does not vanish for any Re_m.

Substituting (10) into (7), one obtains another form of the solution

$$\dot{\delta}_m = \frac{\mathrm{Re}_m^2}{2\pi} \sum_{k=1}^{\infty} \frac{(\dot{f}_m, f_k)}{\lambda_k + \lambda} f_k + j \dot{f}_m \mathrm{Re}_m. \tag{12}$$

If the external magnetic field due to the inductor is represented in the form of a sum of normal travelling and pulsating fields, the right-hand side of Eq. (7) can be written in the form

$$\dot{f}_m(m, r) = \dot{f}_{mt}(m, r) + \dot{f}_{mp}(m, r),$$

where

$$\dot{f}_{mt}(m, r) = -[\omega - v_M(r)]\dot{A}_{0mt}(m, r);$$

$$\dot{f}_{mp}(m, r) = -[\omega - v_M(r)]\dot{A}_{0mp}(m, r);$$

A_{omt} and A_{omp} are the Fourier transforms of the travelling and pulsating components of the vector potential of the external field, and $v_M(r)$ is the fluid velocity relative to the channel walls.

In this case the solution of Eq. (7) can be constructed by putting

$$\dot{\delta}_m = \dot{\delta}_{mt} + \dot{\delta}_{mp},$$

where $\dot{\delta}_{mt}$ and $\dot{\delta}_{mp}$ are found from the equations

$$\dot{\delta}_{mt}(r_M) + j \frac{\mathrm{Re}_m}{2\pi} \int_{R_1}^{R_2} \dot{\delta}_{mt}(r_N) r_N [\omega - m v_M(r_M)] \mathfrak{K}(M, N)\, dr_N = j\,\mathrm{Re}_m \dot{f}_{mt}(r_M);$$

$$\dot{\delta}_{mp}(r_M) + j \frac{\mathrm{Re}_m}{2\pi} \int_{R_1}^{R_2} \dot{\delta}_{mp}(r_N) r_N [\omega - m v_M(r_M)] \mathfrak{K}(M, N)\, dr_N = j\,\mathrm{Re}_m \dot{f}_{mp}(r_M). \tag{13}$$

Applying to (12) the inverse Fourier transform, and separating the real and imaginary parts, we obtain expressions for the active and reactive components of the current density

$$\delta_a(r, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathfrak{Re}\dot{\delta}_m(r, m)\cos(mz) + \mathfrak{Im}\,\dot{\delta}_m(r, m)\sin(mz)]\,dm; \tag{14}$$

$$\delta_r(r, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathfrak{Im}\,\dot{\delta}_m(r, m)\cos(mz) + \mathfrak{Re}\dot{\delta}_m(r, m)\sin(mz)]\,dm;$$

$$\delta(r,z,t)=\sqrt{2}|\dot{\delta}|\cos(\omega t+\psi);\quad \psi=\mathrm{arctg}\,\delta_r/\delta_a.$$

3. We formulate the boundary-value problem for finding the eigenfunctions and characteristic numbers of the operator $\mathfrak{K}$. Performing operations similar to those in the first part of this work, we reach the conclusion that the problem of finding the eigenfunctions f_k and characteristic numbers λ_k is equivalent to the solution of the boundary-value problem

$$\begin{gathered}\Delta_m A_k^+=-\lambda_k\varphi A_k^+;\quad \Delta_m A_k^-=0;\\ A_k^+(R_1)=A_{ki}^-(R_1);\quad A_k^+(R_2)=A_{ke}^-(R_2);\\ \frac{d}{dr}A_k^+(R_1)=\frac{d}{dr}A_{ki}^-(R_1);\quad \frac{d}{dr}A_k^+(R_2)=\frac{d}{dr}A_{ke}^-(R_2);\\ A_{ki}^-(0)<\infty;\quad A_{ke}^-(\infty)=0,\end{gathered}\tag{15}$$

where

$$f_k=\lambda_k\varphi A_k;\quad A_k=\begin{cases}A_k^+(r), & r\in[R_1,R_2],\\ A_{ki}^-(r), & r<R_1,\\ A_{ke}^-(r), & r>R_2.\end{cases}$$

For an arbitrary function $\varphi(r)$ it is impossible to construct an analytic soltuion of problem (15). In this case the eigenfunctions and characteristic numbers can be found by the numerical algorithm described in [3]. If $\varphi(r)$ = const, the functions A_k^+, A_{ki}^-, A_{ke}^- are easily found:

$$\begin{gathered}A_k^+(r)=C_1J_1(\alpha_k r)+C_2N_1(\alpha_k r);\\ A_{ki}^-(r)=C_3I_1(mr);\quad \alpha_k^2=\lambda_k\varphi-m^2;\\ A_{ke}^-(r)=C_4K_1(mr).\end{gathered}\tag{16}$$

Satisfying the boundary conditions and the normalization condition $\|A_k^+\|=1$, we find C_1, C_2, C_3, C_4, α_k,

$$A_k^+(r)=C_1[J_1(\alpha_k r)-a_kN_1(\alpha_k r)];\quad C_1=\|A_k^+\|^{-2};$$

$$a_k=\frac{\alpha_k\left[J_0(\alpha_kR_1)-\frac{J_1(\alpha_kR_1)}{\alpha_kR_1}\right]+\frac{J_1(\alpha_kR_1)}{K_1(mR_1)}\left[K_0(mR_1)+\frac{K_1(mR_1)}{mR_1}\right]}{\frac{N_1(\alpha_kR_1)}{K_1(mR_1)}m\left[K_0(mR_1)+\frac{K_1(mR_1)}{mR_1}\right]+\alpha_k\left[N_0(\alpha_kR_1)-\frac{N_1(\alpha_kR_1)}{\alpha_kR_1}\right]};\tag{17}$$

where J_0, J_1, N_0, N_1, I_0, I_1, K_0, K_1 are Bessel functions, and α_k are the positive roots of

$$\alpha_k\left[J_0(\alpha_kR_2)-\frac{J_1(\alpha_kR_2)}{\alpha_kR_2}\right]-a_k\left[N_0(\alpha_kR_2)-\frac{N_1(\alpha_kR_2)}{\alpha_kR_2}\right]-m\,\frac{J_0(\alpha_kR_2)-a_kN_1(\alpha_kR_2)}{I_1(mR_2)}\left[I_0(\alpha_kR_2)-\frac{I_1(\alpha_kR_2)}{mR_2}\right]=0.$$

From the value of A_k^+ found one easily recovers the normalized eigenfunctions f_k

$$f_k(m,r)=\frac{A_k^+}{\|A_k^+\|}.\tag{18}$$

In a numerical realization of the algorithm described it is necessary to calculate the set of eigenfunctions for different m values, corresponding to approximating the integrals in Eq. (14) by quadrature equations.

If the current density distribution in the channel is known, the force acting on the liquid metal and the magnetic field distribution in the channel can be determined by equations similar to those derived in the first part of this work.

Results of numerical realization of the algorithms suggested in this paper will be presented and discussed in subsequent publications.

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