NATURE OF THE INSTABILITY OF A TURBULENT DYNAMO

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The self-excitation of a magnetic field in a large volume of a uniformly conducting nonmagnetic fluid in intensive motion is produced by an instability of the state of motion without a magnetic field. It is known that this instability begins if the velocity distribution is rather complex and the magnetic Reynolds number is rather large [1].

The instability threshold is determined by investigation of the development of small primers of the magnetic field. It is described kinematically by the dynamo approximation, in which the velocity field $\mathbf{v}(\mathbf{r})$ is assumed to be specified and which is applicable as long as the excited field is small and one can thereby neglect its reverse effect on the velocity distribution, for example, through Lorentzian forces. The problem of a kinematical dynamo is comprised of the linear equations of electrodynamics and the boundary conditions for the electromagnetic field. These equations reduce to the equations of induction and a solenoid nature for the magnetic field:

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\nabla)\mathbf{B} - (\mathbf{B}\nabla)\mathbf{v} = \frac{1}{\mathrm{Rm}} \triangle \mathbf{B}; \qquad \operatorname{div} \mathbf{B} = 0. \tag{1}$$

The requirement that other sources of a field be absent is an essential boundary condition in the dynamo problem. The absence of external fields is characterized by the condition $\mathbf{B}(r \to \infty) = 0$, where r is the distance from the moving region. This condition restricts self-excitation from ordinary amplification of external fields.

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An analogy exists between the two instabilities which appear in the flow of fluid upon an increase in its intensity: between the hydrodynamical instability of laminar flow, which gives rise to turbulence, and the dynamo instability, which gives rise to a magnetoactive state of the flow of an electrically conducting fluid. This analogy is expressed in the similarity of the mathematical formulation of the dynamo problem (1) to the Orr–Sommerfeld equation [2], which describes the development of small perturbations of the velocity **v'**:

$$\frac{\partial \mathbf{v}'}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}'+(\mathbf{v}'\nabla)\mathbf{v}=\frac{1}{\mathrm{Re}}\triangle\mathbf{v}'-\mathrm{grad}\,p'; \qquad \mathrm{div}\,\mathbf{v}'=0, \quad \mathbf{v}'|_{\mathrm{wall}}=0, \tag{2}$$

with a capture condition on the fixed walls. Both systems of equations contain one and the same terms in the unknown functions **B** and **v'**, and each with a single significant dimensionless parameter, $\mathrm{Rm} = \mu_0\sigma vR$ and $\mathrm{Re} = vR/\nu$, respectively. Both problems reduce to a determination of the critical value of the parameter in question – this is the smallest value for which at least one of the characteristic solutions does not die out as $t \to \infty$.

For the helical dynamo model this analogy is expressed in the similarity of the derived solutions to the solutions of hydrodynamical instability of a plane-parallel flow [3]. The analogy between both phenomena permits transferring the whole procedure for investigation applicable in hydrodynamics to the dynamo instability of a helical flow.

The dynamo instability problem for helical models which are axisymmetric and uniform along the symmetry axis [4-6] reduces to the investigation of perturbations of the magnetic field which are representable in the r, φ, z coordinate system in the form of a surface wave:

$$\mathbf{B}(\mathbf{r},t)=(B_r(r),B_\varphi(r),B_z(r))\exp(im\varphi+ikz+pt). \tag{3}$$

The values of the wave characteristics of the magnetic field m, k, and p permitted by Eq. (1) and the boundary conditions are determined by a dispersion equation, which one can represent in general form as

$$F(m,k,p,\mathrm{Rm})=0. \tag{4}$$

The dispersion equation contains quantities as parameters which characterize the specific dynamo model under discussion, i.e., a specified configuration of the velocity field and distribution of the conductivity of the medium.

A procedure for obtaining explicit expressions of Eq. (4) is known for various cases of the stable motion of concentric cylindrical regions [5, 6]. As a result, we obtain transcendental equations whose zeroes can generally be found only numerically.

In this paper we will use the dispersion equation (5) given in [6] in connection with the investigation of the properties of the dynamo instability. It describes a helical model composed of three concentric cylindrical regions: I, $0 < r < r_1$; II, $r_1 < r < r_2$; and III, $r_2 < r < \infty$, between which ideal electrical contact is understood and within whose boundaries the electrical conductivity of the medium σ and the components of the helical velocity $v_r = 0$, $v_\varphi = \omega r$, and $v_z = \omega\varkappa$ are determined by the constants σ^i, ω^i, and $\varkappa^i$, i = I, II, and III. The choice of these constants is arbitrary, and a specific alternative of the model is determined by them. The numerical solution of the dispersion equation for some alternatives of this model which are most interesting from the viewpoint of the nature of the instability observed was performed on a computer with the help of the same programs as in [6].

The zeroes of the dispersion equation determine the temporal development ($\sim \exp pt$) and spatial distribution [$\sim \exp(im\varphi + ikz)$] of all possible modes of the magnetic field. Two different approaches are followed in explaining the presence or absence of an instability. In the first one the increment p in (4) is assumed to be a complex eigenvalue $p = \gamma + i\Omega$ – these are properly the increment γ and frequency Ω of the field. The dispersion equation implicitly determines the dependence of p on the wavenumber k. The latter should be assumed to be real in order to guarantee boundedness of the field in the coordinate z as $|z| \to \infty$. Instability occurs for a given Rm *if there are some eigenvalues for which the increment is not negative* $(\gamma \geq +0)$.

The results of the calculation of the dependence $p = p(k)$ for different Rm are usually represented in the form of curves $\gamma = \mathrm{const}$ and $\Omega = \mathrm{const}$ in the (k, Rm) plane (Fig. 2 in [6]). The neutral curve $\gamma = 0$ divides the (k, Rm) plane into two parts, which correspond to stable $(\gamma < 0)$ and unstable $(\gamma > 0)$ conditions. The point on the neutral curve with the minimum Rm = Rm* determines the critical characteristics: the crit-

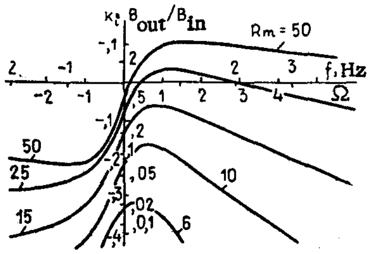

Fig. 1

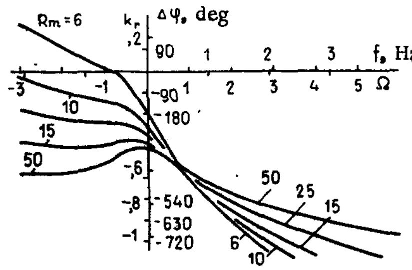

Fig. 2

Fig. 1. Spatial amplification as a function of the field frequency at a finite distance from the source.

Fig. 2. Phase shift of the field along the symmetry axis.

ical magnetic Reynolds number Rm*, the critical frequency Ω^*, and the critical wavenumber k^* of the magnetic field.

This approach corresponds to an investigation of the development in time of nonlocalized perturbations periodic in z and is used most often in connection with the theoretical investigation of hydrodynamical instability. An investigation by this procedure of the dynamo instability of a helical flow was conducted in [5, 6].

However, it is not advantageous in connection with the experimental investigation of a new type of instability to rely only on random perturbations. Below the threshold $Rm < Rm^*$, where there are no phenomena, and even near the threshold $Rm \approx Rm^*$, where the phenomenon is weak and irregular, it is impossible to experimentally predict without a known priming field the distance to the critical values and to adjust a device to the necessary parameters. It is more advantageous to produce perturbations of a strictly controllable quantity and then find the critical conditions from them. It is less convenient in practice to trace the development of perturbations in time; it is more convenient to investigate a steady process and determine the development in space of perturbations which are strictly periodic in the time.

A complex wavenumber $k = k_r + ik_i$ is assumed, in connection with the theoretical investigation of spatial amplification of the field, to be the eigenvalue in the dispersion equation (4), and one should assume p to be purely imaginary $p = i\Omega\ (\gamma \equiv 0)$ — this is the specified frequency of the priming magnetic field. The zeroes of the dispersion equation now determine the function $k = k(\Omega, Rm)$, and it is convenient to present the computational results in the form of $k_r(\Omega)$ and $k_i(\Omega)$ curves, which represent the dependence of the wavenumber and the spatial amplification increment on the frequency of the field for different values of the magnetic Reynolds number.

As an example, the frequency characteristics for the following alternative of a three-region helical model: $r_1 = 1$, $r_2 = 3$, $\sigma^I = \sigma^{II} = 1$, $\sigma^{III} = 0$, $\varkappa^I = 1$, $\omega^{II} = \omega^{III} = 0$, and $Rm = \omega^I \sqrt{2}$ are given in Figs. 1 and 2. The model represents the helical motion of a cylindrical conductor surrounded by a fixed conductor of the same conductivity and a length in the radial direction equal to the diameter of the moving cylinder. The magnetic Reynolds number is defined in terms of the channel radius and the maximum velocity of the motion.

In order to provide a clearer representation of the results, the dimensional quantities normalized to the equipment dimensions [7] are also plotted on the axes of Figs. 1 along with the dimensionless quantities Ω, k_i, and k_r: the cyclical rotation frequency of the magnet of the priming field $f = \Omega/2\pi\mu_0\sigma R^2$ (Hz), the field amplification $B_{out}/B_{in} = \exp(-k_i L/R)$, and the phase difference of the magnetic field waves at the entrance and exit of the device $\Delta\varphi = (k_r L/R)\ (180/\pi)$ (angular degrees). Here $L = 2.2$ m is the height of the device, $R = 0.175$ m is the channel radius, and $\sigma = 6 \times 10^6\ (\Omega \cdot m)^{-1}$ is the conductivity of sodium at 300°C.

The dominant branch of the function $k = k(\Omega)$, which is characterized by the smallest k_i, is given in Figs. 1 and 2. The sign of the frequency Ω determines the direction of rotation of the priming field: a minus sign, in the direction of rotation of the fluid ω, and a plus sign, opposite to ω. It is assumed that one will be able to obtain curves of this type in an experimental model of the helical dynamo [7].

For small Rm the curves in Fig. 1 show attenuation of the applied field along the axis $(-k_i < 0)$. The damping increases with frequency — the curves have a maximum near $\Omega = 0$. As Rm increases, the damping decreases, and for the critical $Rm^* = 20$ at the critical frequency $\Omega^* = 1.0$ the curve reaches the $k_i = 0$

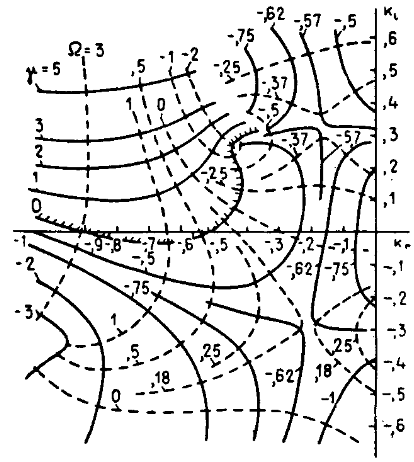

Fig. 3. Saddle point of the function $p = p(k)$ for $Rm = 25 > Rm^*$.

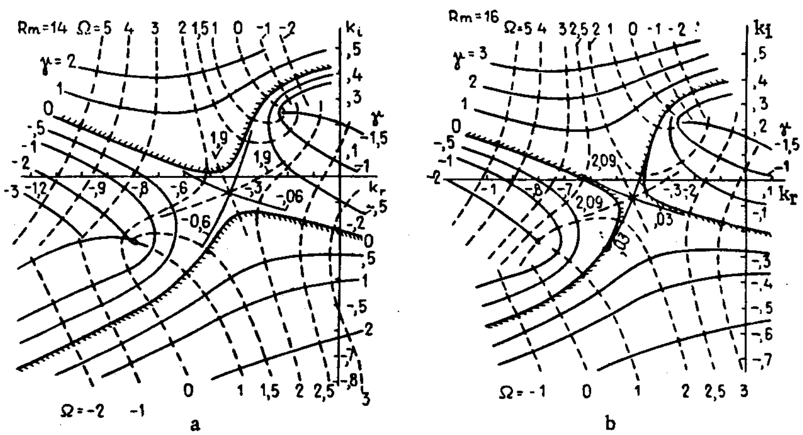

Fig. 4. Neighborhood of the saddle point in the case of counterflow with (a) $Rm < Rm^{**}$ and (b) $Rm > Rm^{**}$.

axis. For $Rm > Rm^*$ the maximum goes beyond the axis, and in a certain frequency interval the field frequency increases downstream $(-k_i > 0)$. The interval of unstable frequencies increases with Rm.

Both approaches in connection with the establishment of the presence of an instability — the investigation of the temporal or spatial development of perturbations — give identical results in the case of neutral perturbations ($\gamma = 0$ and $k_i = 0$), and both are identically effective for determination of the critical characteristics. However, the stability analysis conducted above leaves open the question of the behavior of localized perturbations. In any region fixed relative to z perturbations localized at the initial moment can either increase or tend to zero as time passes. The first case is called absolute instability, and the second, when unstable nonlocalized modes are present, is called convective instability.

In the case of convective instability the wave packet of a perturbation of finite extent both increases with time and moves downstream at a definite group velocity. In a real system of finite length a perturbation can be carried out of an unstable flow region before its amplitude becomes large enough.

In the case of absolute instability the perturbation increases at the point of origin and even moves upstream. As a result perturbations fill the entire flow region, and instability always leads to a new nonlinear pattern of development of the perturbations.

The formulation of this problem in hydrodynamics was accomplished by Landau and Lifshits [2]. Some general statements about the nature of the instabilities are expounded in [3, 8, 9].

According to the notation (3) employed, the group velocity of the perturbations is equal to $v_{gr} = i\partial p/\partial k$. When Rm exceeds Rm^* by just a small amount, the wave packet formed by the unstable $(\gamma \gtrsim 0)$ modes moves

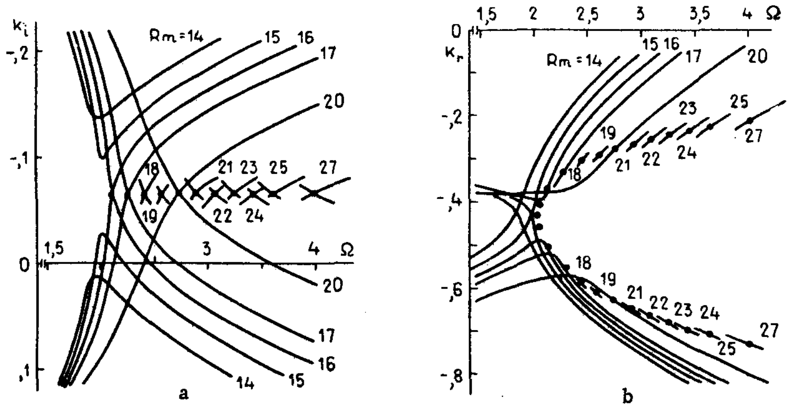

Fig. 5. (a) Spatial amplification of the field and (b) the phase shift in the case of absolute instability; $\varkappa^{II} = \varkappa^{I} = 4/3$, $\omega^{II} = \omega^{I}/3$, and $r_2 = 3r_1$.

at the group velocity, which for $k = k^*$ is equal to $v^*_{gr} = -\partial\Omega/\partial k$. The perturbation would increase at the point of origin if $v^*_{gr} = 0$. It is evident in Fig. 2 that this does not occur. The critical perturbations in helical models with one fixed cylinder move downstream $v^*_{gr} > 0$ (also see [5, Fig. 3; 6, Fig. 2]. The instability when $Rm = Rm^*$ is of the convective nature.

In the general case $\partial p/\partial k$ is a complex quantity, but then the question of the local behavior of the perturbations is determined by the temporal behavior of modes having zero group velocity. As is known (see [2]), one can represent the development in a uniform infinite system of any initially specified perturbation $B(r, \varphi, z, t = 0)$ in the form of a Fourier expansion:

$$\mathbf{B}(z, \varphi, t) = \frac{1}{4\pi^2} \sum_m \int_{-\pi}^{\pi} d\lambda \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} dk \mathbf{B}(\xi, \lambda, 0) e^{ik(z-\xi)+im(\varphi-\lambda)+p_m(k)\cdot t}. \quad (5)$$

The growth or damping as $t \to \infty$ of an arbitrary localized perturbation at any point z is thereby determined by the limit of the integral

$$\lim_{t\to\infty} \sum_m \int_{-\infty}^{\infty} \exp(p_m(k)\cdot t)\, dk, \quad (6)$$

where the dependence $p(k)$ for each m is found by solving the dispersion equation. The integration in (6) is performed over both branches of the function $p(k)$, but in order to determine the presence of an absolute instability, one can restrict oneself to the first branch and the values $Rm \gg Rm^*$, for which the instability interval is found on the real axis k.

The method of steepest descent, by which the problem is reduced to a search for a function $p(k)$ of the saddle points k_s, at which $\partial p/\partial k = 0$, is used to investigate the limit (6). The integral (6) and the local perturbation described by Eq. (5) increase with z fixed if the integration path in (6) passes after its calculation along the outflow lines through saddle points at which $\gamma_s \equiv \mathrm{Re}(p(k_s)) > 0$, i.e., if the modes with complex group velocity equal to zero increase with time. The instability is of the convective nature if all the saddle points in the k plane are located in the region where $\gamma < 0$.

The first branch of the function $p(k)$ is illustrated in Fig. 3 for the model described above. The illustrated solid lines of constant γ are perpendicular to the broken lines of constant Ω, since p is an analytic function of k. One can represent a function which is analytic at the point k_0 in the form of a series:

$$p = p_0 + p'(k-k_0) + \tfrac{1}{2}p''(k-k_0)^2 + \ldots.$$

If $p' \neq 0$, then the Cauchy–Riemann conditions guarantee orthogonality of the level lines of γ and Ω. The coefficient $p' = 0$ at the saddle point k_s. In Fig. 3 these points have the coordinates (−0.21, −0.29) and (−0.16, 0.32). In the neighborhood of k_s the increment $\Delta p = p - p_s$ to the function $p(k)$ is quadratic, and the real part of Δp is constant along the hyperbola $\Delta\gamma = \tfrac{1}{2}p''(\Delta k_r^2 - \Delta k_i^2)$, and the imaginary part is constant along the hyperbola $\Delta\Omega = p''\Delta k_i \Delta k_r$ (to the accuracy of a rotation of the coordinate axes in the case of a complex coefficient p''). There are two curves of the type γ = const and Ω = const near the saddle point which corre-

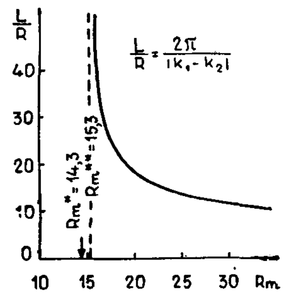

Fig. 6. Length of the globally unstable model according to the points of Fig. 5.

spond to the two branches of these lines.

The first saddle point in Fig. 3 is real — the integration path in the first branch passes through it. However, at both points $\gamma < 0$ (-0.623 and -0.579, respectively). Other saddle points at which $\gamma \geq 0$ would occur have not been detected.

In order to clarify the nature of the instability for still larger Rm, only the curve $\gamma = 0$ in the k plane was calculated with the goal of establishing the passage of the saddle point through this curve. However, in all the investigations of Rm it remains in the region $\gamma < 0$, from which one can conclude that the dynamo instability of a helical model with one fixed element is of the convective nature for Rm > Rm*. An analogous result is known for the hydrodynamical instability of a plane flow.

By modifying the model, one can convert a convective dynamo instability into an absolute one. Removal of the perturbations occurs at the group velocity $v^*_{gr} = -\partial\Omega/\partial k$. At the critical point in Figs. 1 and 2 v^*_{gr} = 4.75, i.e., it amounts to 33% of the value of the critical axial velocity $v_z = Rm(1 + \varkappa^{-2})^{-1/2} = 14.2$. It is sufficient just to bring the medium in region II into a counterflow along a spiral of the same step size as the axial velocity $-v^*_{gr}$ and to lower the velocity in region I just enough so that the fields and currents remain as before, and the removal of perturbations would be completely curtailed in the laboratory reference system. The instability thereby becomes absolute.

In the general case the velocity of the counterflow is not exactly equal to $-v^*_{gr}$, and at first when Rm = Rm* a convective instability arises which develops into an absolute one for Rm = Rm** > Rm*. This is caused by a change in sign of the real part of the function $p = p(k)$ at the saddle point $\partial p/\partial k = 0$ (Fig. 4). The frequency characteristics are constructed from the inverse function $k = k(p)$, for which the point in question is a branch point. When Rm is exactly equal to Rm**, the frequency characteristics $k = k(\Omega)$ and $\gamma = 0$ pass through this branch point and intersect there with the second branch of the solution of the characteristic equation. Until the absolute instability the latter solution corresponds to poles which damp out in the direction of their own propagation (at their own group velocity). For Rm different from Rm**, there is no intersection of the curves $\gamma = 0$ in the complex plane (Fig. 4). For convective instability only the characteristics $k_r(\Omega)$ intersect each other, and for absolute instability — only $k_i(\Omega)$ (Fig. 5). The intersection point of the curves $k_i(\Omega)$, which is marked in Fig. 5 by a dot for all Rm, plays a fundamental role in the mechanism of the absolute instability shown in Fig. 6, since both solutions in it have the identical temporal dependence ($\sim \exp i\Omega t$) and the identical z-dependence of the amplitude ($\sim \exp -k_i z$). Different values of the wavenumbers k_{r_1} and k_{r_2} cause different behavior of the phase of the waves of the excited field, and the phase is shifted by a difference 2π in the distance $L = 2\pi/|k_{r1} - k_{r2}|$.

Division of instabilities into convective and absolute has, just as does the concept of group velocity itself, an exact meaning only in systems which are infinite in z. Near Rm** the length L represents an asymptotic estimate of the minimum length necessary for the onset of instabilities in a bounded system (this is a global instability in the terminology of [9]). Actually, both solutions for Rm** have the identical radial dependence, and one can satisfy the homogeneous zeroth boundary conditions at both end planes $z = 0$ and $z = L$ by a linear combination of them. As one departs from Rm**, differences arise in the radial dependences, and all the remaining solutions of the characteristic equation begin to take part in the satisfaction of the boundary conditions; L becomes only an estimate.

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