MOTION OF AN ELECTRICALLY CONDUCTING LIQUID IN A COLLECTOR SYSTEM WITH ELECTRODYNAMIC COUPLING OF THE INLET AND OUTLET COLLECTORS

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Introduction. The one-dimensional motion of an electrically conducting liquid in collector systems placed in an external magnetic field was examined in [1]. These systems contained an inlet (distributing) and an outlet (receiving) collector and an active section situated between them that consisted of pipes parallel to the magnetic field (Fig. 1). As an example, the hybrid lithium module for the thermonuclear reactor in [2] was built according to this scheme.

In the schemes being examined, the currents flowing in the collectors can be shorted through the electrically conducting liquid moving in the active zone, i.e., there is an electrodynamical coupling of the inlet and outlet collectors. It is of interest to obtain an analytical formula, based on a comparatively simple computational model, that permits estimating the importance of this effect and to carry out an appropriate analysis.

1. Computational System of Equations. In what follows, the analysis is carried out for collector systems of two types, viz., the Π scheme, in which the directions of motion into the inlet and outlet collectors are opposite, and for the Z scheme, in which these directions coincide [1].

We will assume that the extent of the collectors in the direction of the field is small in comparison with the dimensions perpendicular to the field, i.e., the change in the quantities in the direction of flow is slow ($b/L \ll 1$). It is assumed that thin electrically conducting barriers are placed in the collectors in the direction of flow. In addition, $Re_m \ll 1$. These assumptions permit neglecting the $v_y$- and $j_x$-components of the velocity and current density in the collectors and to use the hydraulic approximation for the flow between the electrically conducting barriers (elementary subchannels) [3], i.e., to consider quantities averaged with respect to z.

From the continuity equation for the z-averaged current density, we have

$$b\partial j_{y1}/\partial y = j_{z3}. \tag{1}$$

Here, the indices relate to the corresponding zones (Fig. 1). From Maxwell's equation

$$\oint E\,dl = 0, \tag{2}$$

written for a conductor enclosing the collectors and the active zone, it follows that

$$\int_0^y E_{y1}dy - E_{z3}l + \int_y^0 E_{y2}dy = 0. \tag{3}$$

Here, the contour of integration passes through the points $M_1(x, 0, 0)$, $M_2(x, y, 0)$, $M_3(x, y, l)$, $M_4(x, 0, l)$, and in addition, the condition that the potentials at the points $M_1$ and $M_4$ are equal, which follows from the symmetry of the problem, is used.

Let us further write Ohm's law for the zones:

$$j_{y1} = \sigma(E_{y1} + v_1 B); \quad j_{y2} = \sigma(E_{y2} + v_2 B); \quad j_{z3} = \sigma_3 E_{z3}. \tag{4}$$

Here $v_1$ and $v_2$ are the corresponding velocities in the collectors, $\sigma_3$ is the effective electric conductivity of the active zone in the z direction.

In the y and x directions, electrical conductivity of the active zone vanishes. It follows from the conditions of symmetry that

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$$j_{y1} = -j_{y2}. \tag{5}$$

In addition, it is evident that

$$v_2 = -v_1, \tag{6}$$

$$v_2 = v_{10} - v_1, \tag{7}$$

where $v_{10} = v_1(0, y)$ for the Π and Z schemes, respectively.

After simple transformations, we obtain for the Π and Z schemes, respectively,

$$\frac{\partial^2 j_y^2}{\partial y^2} - \frac{2\sigma_3 j_y}{lb\sigma} + \frac{2\sigma_3 vB}{lb} = 0; \tag{8}$$

$$\partial^2 j_y/\partial y^2 - (2\sigma_3/lb\sigma) j_y + B\sigma_3(2v_1 - v_{10})/bl = 0. \tag{9}$$

Equations (8) and (9), after transformation to dimensionless quantities, are written for the Π and Z schemes in the form

$$\frac{\partial^2 \bar{j}_{y1}}{\partial \bar{y}^2} - k^2 \bar{j}_{y1} + k^2 \bar{v}_1 = 0, \tag{10}$$

$$\frac{\partial^2 \bar{j}_{y1}}{\partial \bar{y}^2} - k^2 \bar{j}_{y1} + k^2(\bar{v}_1 - \bar{v}_{10}/2) = 0 \tag{11}$$

with

$$\bar{j}_{y1} = j_{y1}/\sigma vB; \quad \bar{v}_1 = v_1/v_0; \quad \bar{y} = y/L; \quad k^2 = 2\sigma_3 L^2/lb\sigma.$$

For an elementary subchannel, as in [1], we will write in the hydraulic approximation for quantities averaged with respect to z

$$2\rho v \partial v/\partial x = -\partial p/\partial x - j_y B. \tag{12}$$

In the region parallel to the magnetic field, assuming the flow is laminar, we write:

$$p_1 - p_2 = c\partial v_1/\partial x. \tag{13}$$

Here, the equation of continuity in integral form is used, $b\partial v_1/\partial x = v_{z3}$.

Without the magnetic field

$$v = v_0(l_1 - x)/l_1$$

and, therefore,

$$\Delta p_0 = -v_0 c/l_1, \tag{14}$$

where $v_0$ is the velocity at the collector inlet.

After transforming to dimensionless quantities, we obtain

$$2R\bar{v}_1 \frac{\partial \bar{v}_1}{\partial \bar{x}} - \frac{\partial \bar{p}_1}{\partial \bar{x}} - \frac{\bar{j}_{y1} N^2}{2} = 0; \quad \bar{p}_1 - \bar{p}_2 = \Delta\bar{p} = -\frac{\partial \bar{v}_1}{\partial \bar{x}}; \tag{15}$$

$$\bar{x} = x/l_1; \quad R = -\rho v_0 l_1/c; \quad N^2 = -2\sigma B^2 l_1^2/c; \quad \Delta\bar{p} = (p_1 - p_2)/\Delta p_0.$$

The base pressure $\Delta p_0$ and velocity $v_0$ are related by relationship (14) and, in addition, one of these quantities is considered to be given. The parameter N denotes Hartmann's number and the parameter R denotes Reynolds' number.

2. Π Scheme. Let us next examine the Π scheme. We will write the difference in pressure between the inlet and outlet collector systems in the form

$$\bar{p}_1(0) - \bar{p}_2(0) = (\bar{p}_1(0) - \bar{p}_1(\bar{x})) + (\bar{p}_1(\bar{x}) - \bar{p}_2(\bar{x})) + (\bar{p}_2(\bar{x}) - \bar{p}_2(0)) = \Delta\bar{p}. \tag{15a}$$

Substituting into expression (15a) the functions (5), (6), and (15), we obtain:

$$\Delta\tilde{p}=-\frac{\partial\bar{v}_1}{\partial\bar{x}}+\int_0^{\bar{x}}\bar{j}_{y1}N^2 d\bar{x}. \quad (16)$$

Differentiating Eq. (16) with respect to x, we obtain an equation for the motion in the inlet collector in the form

$$\partial^2\bar{v}_1/\partial\bar{x}^2-N^2\bar{j}_{y1}=0. \quad (17)$$

With the help of (16) we obtain the boundary conditions for the case when the pressure at the inlet into the collector system and the outlet from it are uniform in the form

$$-\Delta\tilde{p}=\left.\frac{\partial\bar{v}_1}{\partial\bar{x}}\right|_{\bar{x}=0}; \quad (18)$$

$$\bar{v}_1(1,\bar{y})=0. \quad (19)$$

In addition, it is obvious that

$$\bar{j}_y\big|_{\bar{y}=0.5}=0; \quad \partial\bar{j}_y/\partial\bar{y}\big|_{\bar{y}=0}=0. \quad (20)$$

The problem reduces to solving the system of equations (10) and (17) with boundary conditions (18), (19), and (20). We seek the solution in the form of a series with respect to the eigenfunctions of the equation $z'' + N^2z' = 0$ with boundary conditions (20), i.e., in the form

$$\bar{v}_1=\sum_{n=0}^{\infty}v_n(\bar{x})\cos\mu_n\bar{y}; \quad \bar{j}_{y1}=\sum_{n=0}^{\infty}j_n(\bar{x})\cos\mu_n\bar{y} \quad (\mu_n=2\pi(n+1/2)). \quad (21)$$

The boundary conditions (20) are thereby satisfied.

Let us expand unity in terms of the series

$$1=\sum_{n=0}^{\infty}M_n\cos\mu_n\bar{y}; \quad M_n=\frac{4(-1)^n}{\mu_n}. \quad (21a)$$

This series will be needed in what follows. After substituting the series (21) into Eqs. (10) and (17), we obtain:

$$j_n=v_n a_n^2; \quad (22)$$

$$d^2v_n/d\bar{x}^2-r_n^2v_n=0. \quad (23)$$

Here,

$$a_n^2=k^2/(\mu_n^2+k^2); \quad r_n=Na_n. \quad (24)$$

Substituting expression (21) into the boundary conditions (18) and (19), using at the same time the expansion (21a), we write the boundary conditions for Eq. (23):

$$v_n(1)=0; \quad dv_n/d\bar{x}\big|_{\bar{x}=0}=-M_n\Delta\tilde{p}. \quad (25)$$

We now obtain the solution of Eq. (23) in the form

$$v_n=\Delta\tilde{p}\left(e^{r_n(2-\bar{x})}-e^{r_n\bar{x}}\right)M_n/(1+e^{2r_n})r_n. \quad (26)$$

For the velocity, we have:

$$\bar{v}_1=\Delta\tilde{p}\sum_{n=0}^{\infty}\left(e^{r_n(2-\bar{x})}-e^{r_n\bar{x}}\right)M_n\cos\mu_n\bar{y}/(1+e^{2r_n})r_n. \quad (27)$$

We write the flow rate in the form

$$G=2\int_0^{0.5}\bar{v}_1 d\bar{y}=2\sum_{n=0}^{\infty}\frac{v_n(-1)^n}{\mu_n}=\Delta\tilde{p}\sum_{n=0}^{\infty}\frac{8\left(e^{r_n(2-\bar{x})}-e^{r_n\bar{x}}\right)}{r_n\mu_n^2(1+e^{2r_n})}. \quad (28)$$

The total flow rate through the collector system equals

$$\bar{G}_0=\Delta\bar{p}\sum_{n=0}^{\infty}\frac{8(e^{2r_n}-1)}{r_n\mu_n^2(1+e^{2r_n})}=\Delta\bar{p}f. \tag{29}$$

If the pressure differentials between the collectors are maintained constant with a change in the magnetic field, then, choosing $\Delta p_0$ as the pressure differential on the collector, we can set $\Delta\tilde{p} = 1$ in expressions (27) and (28). In the case of constant flow rates, we will take the average velocity at the collector inlet for $v_0$; then $\bar{G}_0 = 1$ and from expression (29) it follows that

$$\Delta\bar{p}=f^{-1}. \tag{30}$$

*Let us now examine the case when the velocity distribution is uniform at the inlet to* the collector system and does not depend on the magnetic field. Choosing as $v_0$ the velocity at the collector inlet we obtain

$$v_1(1,\bar{y})=0;\quad v_1(0,\bar{y})=1. \tag{31}$$

Substituting expression (21) into the boundary conditions (31), making use of the expansion (21a) at the same time, we have

$$\bar{v}_{1n}(1)=0;\quad \bar{v}_{1n}(0)=M_n. \tag{32}$$

Solving Eq. (23) with boundary conditions (32), after substituting the result into expression (21), we obtain:

$$\bar{v}=\sum_{n=0}^{\infty}\frac{4(-1)^n}{\mu_n}\cos\mu_n\bar{y}\,\frac{e^{r_n(2-\bar{x})}-e^{r_n\bar{x}}}{e^{2r_n}-1}. \tag{33}$$

<u>3. Z Scheme.</u> In this case, we write

$$\bar{p}_1(0)-\bar{p}_2(0)=(\bar{p}_1(0)-\bar{p}_1(\bar{x}))+(\bar{p}_1(\bar{x})-\bar{p}_2(\bar{x}))+(\bar{p}_2(\bar{x})-\bar{p}_2(1))=\Delta\bar{p}. \tag{34}$$

Taking into account the functions (5) and (7), with the help of expressions (15), we obtain:

$$\Delta\bar{p}=-\int_0^{\bar{x}}\left(2R\bar{v}_1\frac{\partial\bar{v}_1}{\partial\bar{x}}+\frac{\bar{j}_{y1}N^2}{2}\right)d\bar{x}-\frac{\partial\bar{v}_1}{\partial\bar{x}}+\int_1^{\bar{x}}\left(2R(\bar{v}_{10}-\bar{v}_1)\frac{\partial\bar{v}_1}{\partial\bar{x}}+\frac{\bar{j}_{y1}N^2}{2}\right)dx. \tag{35}$$

Differentiating (35) with respect to $\bar{x}$, we arrive at the equation of motion in the inlet collector:

$$\partial^2\bar{v}_1/\partial\bar{x}^2-2R\bar{v}_{10}\partial\bar{v}_1/\partial\bar{x}-\bar{j}_yN^2=0. \tag{36}$$

Let us now limit ourselves to the case when the velocity distribution at the collector inlet is uniform and does not depend on the magnetic field.

Using for $\bar{v}_0$ the velocity at the collector inlet, we obtain $\bar{v}_{10} = 1$. The equation of motion is written in the form

$$\partial^2\bar{v}_1/\partial\bar{x}^2-2R\partial\bar{v}_1/\partial\bar{x}-\bar{j}_yN^2=0. \tag{37}$$

Its boundary conditions are

$$\bar{v}_1(0,y)=1;\quad \bar{v}_1(1,y)=0. \tag{38}$$

The equation for the current density is written in the form

$$\partial^2 j_{y1}/\partial\bar{y}^2-k^2\bar{j}_{y1}+k^2\bar{v}_1=k^2/2. \tag{39}$$

We seek the solution in the form of the series (21). After substituting them into Eqs. (37) and (39) with the boundary conditions we obtain:

$$j_{yn}=a_n^2(v_n-M_n/2);$$

$$d^2v_n/d\bar{x}^2-2R\,dv_n/d\bar{x}-v_nN^2a_n^2+N^2a_n^2M_n/2=0; \tag{40}$$

$$v_n(1)=0; \quad v_1(0)=M_n. \tag{41}$$

Solving Eq. (40) with boundary conditions (41) and substituting the results into (21), we obtain:

$$\bar{v}_1=\sum_{n=0}^{\infty}\left(\frac{e^{r_1\bar{x}}-e^{r_2\bar{x}}}{1-e^{r_1-r_2}}+e^{r_2\bar{x}}\right)M_n\cos\mu_n\bar{y} \qquad (r_{1,2}=R\pm\sqrt{R^2+a_n^2N^2}). \tag{42}$$

4. Smoothing the Flow Rates. Let us assume that in the Π scheme, with uniform pressure at the inlet and outlet collector system, a uniform velocity distribution is created in the active zone with the help of a throttling array, placed at the inlet in the pipes parallel to the magnetic field. In this case,

$$p_1(x)-p_2(x)=(c+\delta(x,y))\,\partial v/\partial x, \tag{43}$$

where $\delta(x, y)$ is the resistance contributed by the throttling array. In dimensionless form

$$\bar{p}_1(\bar{x})-\bar{p}_2(\bar{x})=-(1+\bar{\delta})\,\partial\bar{v}/\partial\bar{x}. \tag{44}$$

Using expressions (15), (15a), (5), and (6) we write:

$$\Delta\bar{p}=-(1+\bar{\delta})\frac{\partial\bar{v}_1}{\partial\bar{x}}+\int_0^{\bar{x}}\bar{j}_{y1}N^2d\bar{x}. \tag{45}$$

For fixed average velocity at the inlet to the collector system, taken as the base velocity,

$$\bar{v}_1=1-\bar{x}. \tag{46}$$

For the current density, we obtain with the help of (10)

$$\partial^2\bar{j}_{y1}/\partial\bar{y}^2-k^2\bar{j}_{y1}+k^2(1-\bar{x})=0. \tag{47}$$

From here, taking into account the boundary conditions (20)

$$\bar{j}_y=(1-\bar{x})\left(1-\frac{\operatorname{ch}ky}{\operatorname{ch}k/2}\right). \tag{48}$$

Assuming that $\delta(1, 0) = 0$, i.e., there is no resistance contributed at the pipe inlet with minimum flow rate, with the help of (45) and (46) we obtain:

$$\Delta\bar{p}=1+\frac{N^2}{2}\left(1-\frac{1}{\operatorname{ch}k/2}\right). \tag{49}$$

At the same time, it follows from (45) that

$$\bar{\delta}=\frac{N^2}{2}\left(1-\frac{1}{\operatorname{ch}k/2}\right)-\frac{N^2(\bar{x}-\bar{x}^2)}{2}\left(1-\frac{\operatorname{ch}k\bar{y}}{\operatorname{ch}k/2}\right). \tag{50}$$

We will now likewise clarify the possibility of partial smoothing of flow rates with uniform increase in the resistance of the throttling array from the value c to c'. In this case, evidently,

$$c'=cN^2/N_0^2, \tag{51}$$

where $N_0$ is the value of the dimensionless parameter for which the degree of nonuniformity in the velocity distribution in the active zone is small, and N is the instantaneous value of this parameter.

With the help of relations (30), (14), and (51) we obtain

$$\Delta\bar{p}=\frac{1}{f(N_0)}\frac{N}{N_0^2}. \tag{52}$$

5. Computational Results. In order to improve the convergence of the series in formulas (27) and (29), we will rewrite them in the form

$$\bar{v}_1=1-\bar{x}+\sum_{n=0}^{\infty} M_n \cos\mu_n\bar{y}\left[\frac{e^{r_n(2-\bar{x})}-e^{r_n\bar{x}}}{r_n(1+e^{2r_n})}-1+\bar{x}\right]; \quad (27a)$$

$$\bar{G}_0=1+8\sum_{n=0}^{\infty}\left[\frac{e^{2r_n}-1}{r_n(1+e^{2r_n})\mu_n^2}-\frac{1}{\mu_n^2}\right]=\Delta\bar{p}f. \quad (29a)$$

The calculations were performed for the geometry of the lithium module for the thermonuclear reactor described in [2], i.e., for the Π scheme. The velocity distributions in the inlet collector are shown in Figs. 2 and 3.

For large N, the velocity distribution is not uniform in the cross section of the collector as well as along its length. From Fig. 2, it is evident that in the center of the collector at its inlet, the liquid is decelerated and this leads to the formation of M-shaped velocity profiles. In sections of the collector farther away from the inlet, back flows and corresponding circulatory flows arise at the center of the collector in the active zone (Fig. 3). As the parameter k decreases by a factor of 2, which is equivalent to placing a single insulating barrier in the collectors in the x0z plane, the nonuniformity of the velocity distribution decreases (dashed lines in Figs. 2 and 3).

The dependence of the pressure differential on the parameter N is shown in Fig. 4. As N increases, the pressure differential increases. As the velocity distribution becomes less nonuniform due to insertion of the insulating barrier, pressure losses decrease at the same time. The pressure losses increase with the use of a throttling array at the inlet to the active zone in order to smooth the velocity profiles.

From Fig. 3 it is evident that with incomplete smoothing of flow rates, which does not require adjusting the spatial distribution of the resistance of the throttling array, the pressure losses can be less than with the more difficult to realize complete equalization of flow rates.

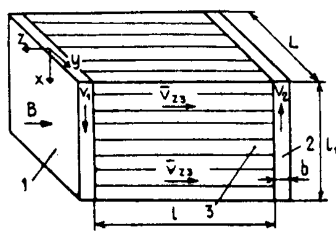


Fig. 1. Hydraulic scheme for a blanket module for a thermonuclear reactor. 1) Inlet collector; 2) outlet collector; 3) active region.

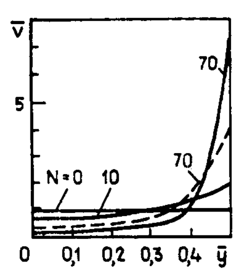


Fig. 2. Velocity distribution over the cross section of the inlet collector at its inlet. The solid lines correspond to $k^2$ = 0.43, x = 0; the dashed lines correspond to $k^2$ = 0.108, x = 0.

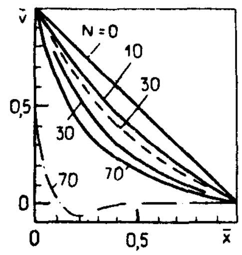


Fig. 3

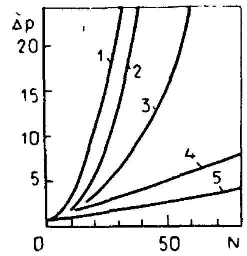


Fig. 4

Fig. 3. Velocity distribution along the length of the inlet collector. The solid lines show the velocity averaged over the cross section of the collector with $k^2$ = 0.43; dashed lines show the velocity averaged over the cross section of the collector with $k^2$ = 0.108; the dot-and-dash line corresponds to the velocity along the collector axis with $k^2$ = 0.43.

Fig. 4. Pressure differential on the collectors as a function of the parameter N. 1) $k^2$ = 0.43, uniform flow, throttling; 2) $k^2$ = 0.43, incomplete smoothing of flow rates; 3) $k^2$ = 0.108, uniform flow, throttling; 4) $k^2$ = 0.43, natural flow; 5) $k^2$ = 0.108, natural flow.

Conclusions. 1. Analytical functions have been obtained for estimating the effect of electrodynamic coupling between the inlet and outlet collectors in collector systems of the Z and Π type placed in a magnetic field.

2. The velocity profiles in the collector systems are substantially nonuniform. In the Π scheme, M-shaped velocity profiles arise in the collectors; back flows are possible in the collector, and closed circulatory flows in the active zone.

3. The velocity distribution in the collector system can be smoothed either by placing insulating barriers, which decreases pressure losses, or placing a throttling array at the inlet to the active zone, which increases the pressure differential.

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