TURBULENT FLOW OF AN ELECTRICALLY CONDUCTING FLUID BETWEEN PARALLEL PLATES IN A TRAVELING MAGNETIC FIELD

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Introduction. Turbulent flow in a traveling magnetic field has been investigated in a number of studies [1-4]. It is shown in [1] that for a small variation of the induction along the gap height (gap small compared to the pole division and large compared to the depth of penetration of the field) the dependence of the hydraulic drag coefficient on the nondimensional similarity criteria, i.e., Reynolds number Re and Stewart number St, is the same as for a constant field. The Stewart number is computed from the effective value of induction of the total magnetic field and the velocity of the electrically conducting fluid. It is noted in [2] that the analysis carried out in [1] does not take account of the motion of the field relative to the fluid with required degree of completeness; in view of this a modified Stewart number $St_M = Sts/(1 - s)$ is introduced. In [3] a range of variation of the parameters is determined, in which the viscous dissipation in the channel is the same as in the absence of the field, which leads to a dependence of the hydraulic drag coefficient of the same form as in [1] for plane flow. In [4] a correction to the shearing stress tensor [1] is obtained which depends on the ratio of the time scale of the turbulence to the period of the field.

In the present article the problem of computing the dependence of the hydraulic drag coefficient on the governing parameters is formulated based on the solution of differential equation of motion of Reynolds.

1. System of Equations and Algorithm for Computation. A schematic of the flow is shown in Fig. 1. The external magnetic field is given in the form [5, p. 322]

$$H^0{}_x = -iH_m \operatorname{sh} \alpha z \cdot e^{i(\omega t - \alpha x)}, \quad H^0{}_y = 0, \quad H^0{}_z = H_m \operatorname{ch} \alpha z \cdot e^{i(\omega t - \alpha x)}.$$

Here H_m is the amplitude of the intensity of the external magnetic field at z = 0; $\alpha = \pi/\tau$; τ is the pole division; ω is the frequency of the traveling field. The velocitity vector is of the form v = (u(z),0,0). We write the induced field $h = (h_x, 0, h_z)$ in the form of a wave traveling along the x axis:

$$h_x = \dot{h}_x e^{i(\omega t - \alpha x)}, \quad h_z = \dot{h}_z e^{i(\omega t - \alpha x)}.$$

Here $\dot{h}_x(z)$, $\dot{h}_z(z)$ are the complex amplitudes of the induced field intensity. The projection of the induction equation, written in nondimensional form, on the z axis gives [5, p. 323]:

$$d^2\bar{h}/d\bar{z}^2 - q^2\{\bar{h} + iR(C - \bar{u})[\operatorname{ch} q\bar{z} + \bar{h}]\} = 0, \quad \bar{z} = \frac{z}{\delta} \quad \bar{h} = \frac{h_z}{H_m}, \quad \bar{u} = \frac{u}{U_c}, \tag{1}$$

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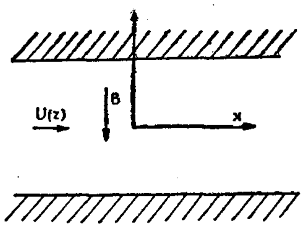

Fig. 1. Schematic of the flow.

$$U_c=\frac{1}{2\delta}\int\limits_{-\delta}^{\delta}u(z)\,dz,$$

$$C=\frac{\omega}{\alpha U_c}=\frac{1}{1-s}=\frac{u_s}{U_c},$$

$$q=\alpha\delta,\quad R=\bar{\omega}(1/s-1)/q^2,\quad \bar{\omega}=\mu\sigma\omega s\delta^2,$$

s is the slip, σ is the electrical conductivity, and u_s is the velocity of the traveling field.

Making use of div **h** = 0 and the condition of perpendicularity of the lines of force to the ferromagnetic wall of the channel, we get

$$z=\pm\delta,\quad \partial h_z/\partial z=-\partial h_x/\partial x=i\alpha h_x=0.$$

Therefore, for Eq. (1) we have the boundary condition

$$d\bar{h}/d\bar{z}=0,\quad \bar{z}=1. \tag{2}$$

Form the condition of symmetry we have

$$d\bar{h}/d\bar{z}=0,\quad \bar{z}=0. \tag{3}$$

Let us write the equation of motion. For this purpose we must determine the electromagnetic force density $\langle\mathbf{f}\rangle$ averaged over a period of the field. As usual we put

$$\langle\mathbf{f}\rangle={}^1/_2\,\Re\{\mathbf{j}\times\mathbf{B}^*\}. \tag{4}$$

Making use of $\mathbf{j}$ = curl **h** and (1) we get

$$\langle\mathbf{f}\rangle_x=\frac{1}{2}\frac{\sigma\omega B_m^2}{\alpha}\left(1-\frac{\alpha u}{\omega}\right)|\bar{H}|^2. \tag{5}$$

Here $|\bar{H}|^2=(\mathrm{ch}\,q\bar{z}+h_r)^2+h_\alpha^2$ is the squared modulus of the amplitude of the total magnetic field), $B_m=\mu H_m$, $\bar{h}=h_r+ih_\alpha$.

We write the projection of the equation of motion on the x axis in the form

$$d\tau_{xz}/dz+\langle\mathbf{f}\rangle_x-\partial p/\partial x=0.$$

Here $\partial p/\partial x$ is the pressure gradient; $\tau_{xz}=\tau_l+\tau_t$ is the total shearing stress due to friction; $\tau_l=\eta du/dz$, $\eta=\nu\rho$ is the dynamic viscosity coefficient; $\tau_t=-\rho\langle u'w'\rangle$.

For the turbulent stress we use the Prandtl formula $\tau_t=\rho l^2|du/dz|du/dz$, where l is the length of the displacement path. Using (5) and reducing the equation of motion to the nondimensional form, we get

$$\frac{d}{d\bar{z}}\left[\left(\frac{1}{\mathrm{Re}}+\bar{l}^2\left|\frac{d\bar{u}}{d\bar{z}}\right|\right)\frac{d\bar{u}}{d\bar{z}}\right]+\mathrm{St}(C-\bar{u})|\bar{H}|^2+\frac{\lambda}{2}=0. \tag{6}$$

Here Re = $\delta U_c/\nu$ is Reynolds number; St = $\frac{1}{2}\sigma B_m^2 \delta/\rho U_c$, Stewart number computed from the effective value of the external field; $\lambda = -(\partial p/\partial x)\delta/(\rho U_c^2/2)$, nondimensional pressure gradi- The boundary conditions for the velocity are:

$$\bar{u}(1) = d\bar{u}/d\bar{z}(\bar{z}=0) = 0.$$

Equation (6) differs from the corresponding equation of motion in a constant field particularly in the fact that the integral of the electromagnetic volume force over the entire depth of the channel does not vanish and, therefore, there is no explicit relation between λ and the frictional stress at the wall of the channel.

Integrating (6) from 0 to $\bar{z}$ and introducing the new variable

$$\bar{v}(\bar{z}) = \mathrm{St}\int_0^{\bar{z}} (C-\bar{u})\,|\bar{H}|^2 d\bar{z},$$

we obtain a system of equations equivalent to (6):

$$\varepsilon(\bar{z})\frac{d\bar{u}}{d\bar{z}} + \bar{v} + \frac{\lambda\bar{z}}{2} = 0;\quad \frac{d\bar{v}}{d\bar{z}} - \mathrm{St}(C-\bar{u})\,|\bar{H}|^2 = 0. \tag{7}$$

Here we have used the notation

$$\varepsilon(\bar{z}) = 1/\mathrm{Re} + \bar{l}^2\,|d\bar{u}/d\bar{z}|. \tag{8}$$

The boundary conditions are:

$$\bar{u}(1) = \bar{v}(0) = 0. \tag{9}$$

We determine coefficient λ from the normalization condition:

$$\int_0^1 \bar{u}(\bar{z})\,d\bar{z} = 1. \tag{10}$$

For the path of displacement $\bar{l}$ we take the dependence used in the case of constant field [6, p. 180]:

$$\bar{l} = \bar{l}_0/(1+\varkappa(N\bar{l}_0)^{3/2}).$$

We shall compute the Stewart number occurring in the expression for $\bar{l}$ from the effective value of the total local field. We write

$$\bar{l} = \bar{l}_0/(1+\varkappa(\mathrm{St}\,|\bar{H}|^2\bar{l}_0)^{3/2}). \tag{11}$$

As in [6], $\bar{l}_0 = D\bar{l}_{00}$, $\bar{l}_{00} = 0.14 - 0.08(1-\bar{z}_W)^2 - 0.06(1-\bar{z}_W)^4$, $\quad = 10^5$, $\bar{z}_W$ is the dimensionless distance to the channel wall.

Van Dreist factor is [6]

$$D = 1 - \exp\left(-z_W\sqrt{\tau_W/\rho}/\nu A\right),$$

where $\tau_W = \eta du/dz$ at $z = \delta$, $A = 27.5$.

Taking the first equation of system (7) into consideration, we can write

$$D = 1 - \exp\left(-\bar{z}_W\,\mathrm{Re}\sqrt{|\bar{v}(1) + \lambda/2|}/A\right). \tag{12}$$

Here $\bar{v}(1)$ is the value of the function $\bar{v}(\bar{z})$ at the wall of the channel obtained from the solution of (7).

The problem now reduces to integration of the system of equations (1), (7) with boundary conditions (2), (3), (9) and normalization condition (10).

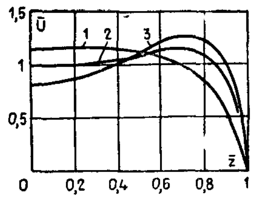

Fig. 2

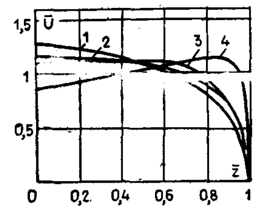

Fig. 3

Fig. 2. Velocity distribution over the gap for Re = 3000, $q = 0.1$, $\lambda_H = 2.5 \cdot 10^{-3}$; $\bar{\omega}$ and s: 1) 2 and 0.7; 2) 10 and 0.3; 3) 2 and 0.9.

Fig. 3. Velocity distribution over the gap for Re = 3000, $q = 0.5$, $\bar{\omega} = 0.01$, $s = 0.3$; λ_H: 1) 0.002; 2) 0; 3) 0.004; 4) 0.021.

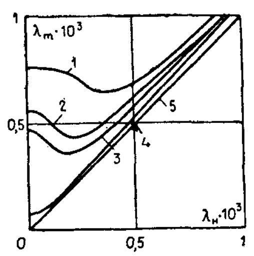

Fig. 4

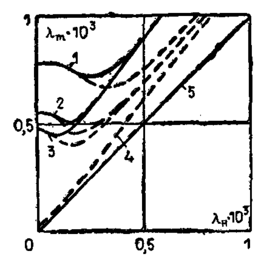

Fig. 5

Fig. 4. Dependence of the hydraulic drag coefficient λ_m on parameter λ_H for $\bar{\omega} = 0.01$, $q = 0.1$, $s = 0.3$–0.7; Re: 1) $3 \cdot 10^3$; 2) 10^4; 3) $2 \cdot 10^4$; 4) laminar flow; 5) graph of $\lambda_m = \lambda_H$.

Fig. 5. Dependence of the hydraulic drag coefficient λ_m on parameter λ_H for $\bar{\omega} = 0.1$, $q = 0.5$; Re: 1) $3 \cdot 10^3$; 2) 10^4; 3) $2 \cdot 10^4$; continuous curves $s = 0.7$; dashes $s = 0.3$; 4) laminar flow; 5) graph of $\lambda_m = \lambda_H$.

2. Similarity Criteria. Below we shall compute the hydraulic drag coefficient $\lambda_m = \tau_w / (\rho U_c^2/2)$ as a function of the dimensionless quantities (similarity criteria) q, $\bar{\omega}$, s, Re introduced above, and the quantity $\lambda_H = 2\text{Ha}_{\text{eff}}/\text{Re}$. Here $\text{Ha}_{\text{eff}} = B_m \delta \sqrt{\sigma / 2\rho\nu (1 + \text{Rm}^2)}$ is the Hartmann number computed from the effective value of the total field induction in the channel: $\text{Rm} = \mu\sigma(u_s - U_c)\tau/\pi$ is magnetic Reynolds number computed from the size of the pole division τ. It is obvious that $\text{St} = \lambda_H^2 \text{Re}(1 + (\bar{\omega}/q^2)^2)$. Let us make some clarifications here. When the parameter $q = \alpha\delta$, which characterizes the geometry of the channel, is small, the change in the external magnetic field over the height of the gap is negligible. Parameter $\bar{\omega}$ is essentially the magnetic Reynolds number computed over the half-height of the gap δ and the relative velocity $u_s - U_c$. Furthermore, $\bar{\omega} = 2(\delta/h)^2$, where $h = \sqrt{2/\mu\sigma\omega s}$ is the depth of penetration of the plane electromagnetic wave into the half-space. Therefore, for small values of $\bar{\omega}$ the field and currents induced in the electrically conducting fluid are homogeneous over the gap. Parameter λ_H equals the hydraulic drag coefficient of laminar flow in a constant magnetic field at large Hartmann numbers with induction equal to the effective value of the total induction field of the investigated flow. It is a convenient parameter to use, since firstly, for small q and $\bar{\omega}$ it may be expected [1] that the hydraulic drag coefficient will be independent of the slip, and secondly, for small q and $\bar{\omega}$, $\lambda_m \to \lambda_H$ [1] for all Re.

3. Algorithm and Results of Computation. We give a brief description of the algorithm for the solution. It consists of two subprograms. The first enables us to determine the average electromagnetic force over the cross section from the solution of induction equation (1) with boundary conditions (2), (3) and specified velocity distribution $\bar{u}(\bar{z})$. The equation was solved by the standard difference method using trial run algorithm.

The second subprogram enables us to solve the system of equations (7) with boundary conditions (9) and normalization condition (10) for a given distribution of the electromagnetic force.

A nonuniform grid $0 = \bar{z}_1 < \bar{z}_{N+1} = 1$ was constructed for solving system (7); the grid became denser near the channel wall, the degree of densening was controlled by the value of the gradient $d\bar{u}/d\bar{z}$ at $\bar{z} = 1$. An auxiliary variable

$$t = \ln\left(1 + 0.5\,\mathrm{Re}\,\sqrt{\bar{\tau}_W}\,(1 - \bar{z})\right)$$

was introduced for this purpose; here $\bar{\tau}_W = \tau_W/\rho U_c^2$ is some approximate value of the nondimensional frictional stress at the wall of the channel.

A uniform grid with step $\Delta t = 0.02\ \ln(1 + 0.5\ \mathrm{Re}\ \sqrt{\bar{\tau}_W})$ was constructed in the segment $\Delta t = 0.02 \ln(1 + 0.5\,\mathrm{Re}\,\sqrt{\bar{\tau}_W})$. The nonuniform grid in the segment $\bar{z} \in (0, 1)$ is obtained from the formula [9]

$$\bar{z}_i = 1 - \frac{2}{\mathrm{Re}\,\sqrt{\bar{\tau}_W}}\,(e^{t_i} - 1).$$

On each segment $[\bar{z}_i, \bar{z}_{i+1}]$ of the wall, system (7) was replaced by an approximate system with constant coefficients

$$\varepsilon_{i+1/2}\frac{d\bar{u}}{d\bar{z}} + \bar{v} + \frac{\lambda\bar{z}}{2} = 0;\quad \frac{d\bar{v}}{d\bar{z}} - \mathrm{St}\,(C - \bar{u})\,|\bar{H}_{i+1/2}|^2 = 0. \tag{7a}$$

Here

$$\varepsilon_{i+1/2} = (\varepsilon_i + \varepsilon_{i+1})/2,\quad |\bar{H}_{i+1/2}|^2 = (|\bar{H}_i|^2 + |\bar{H}_{i+1}|^2)/2.$$

The general solution on segment $|\bar{z}_i, \bar{z}_{i+1}|$ is

$$\bar{u}_i(\bar{z}) = A_i e^{p_i\bar{z}} + B_i e^{-p_i\bar{z}} + \frac{\lambda}{\mathrm{St}\,|\bar{H}_{i+1/2}|^2} + C;$$

$$\bar{v}_i(\bar{z}) = -\varepsilon_{i+1/2}\,p_i\,(A_i e^{p_i\bar{z}} - B_i e^{-p_i\bar{z}}) - \frac{\lambda\bar{z}}{2};$$

$$p_i = |\bar{H}_{i+1/2}|\sqrt{\mathrm{St}/\varepsilon_{i+1/2}};$$

A_i, B_i are arbitrary constants. From the condition of continuity of the functions $\bar{u}(\bar{z})$, $\bar{v}(\bar{z})$ at the grid nodes $\bar{z}_i$, $\bar{u}_i(\bar{z}_i) = \bar{u}_{i+1}(\bar{z}_i)$, $\bar{v}_i(\bar{z}_i) = \bar{v}_{i+1}(\bar{z}_i)$, $i = 2,, \ldots, N$ and from the boundary conditions $\bar{v}_1(0) = \bar{u}_N(1) = 0$ we obtain a system of two-point equations for determining A_i, B_i. The integral in (10) was replaced by the sum of integrals over the segments $[\bar{z}_i, \bar{z}_{i+i}]$. Within each segment function $\bar{u}(\bar{z})$ was replaced by $\bar{u}_i(\bar{z})$, for which the integral was computed analytically. The system of equations for A_i, B_i, and λ thus obtained was solved by the method of orthogonal trial run [10, p. 112]. The common solution of system (1) and (7) was obtained by the method of successive approximations.

Examples of the velocity distributions over the gap, computed from the solution of the system of equations (1), (7), are shown in Figs. 2 and 3. As in the case of laminar flow, the formation of M-shaped velocity profiles is observed. This effect gets enhanced with the increase of the slip and the parameters $\bar{\omega}$ and q (Fig. 2). An example of the change of the velocity profile with the increase of parameter λ_H from zero onward is shown in Fig. 3. At first there is a partial suppression of the velocity oscillations by the magnetic field. The fluid is accelerated in the central region of the channel and decelerated near the wall. Later velocity profile gets flattened again due to the Hartmann effect. A further increase in λ_H results in the formation of the M-shaped velocity profile.

The dependence of the hydraulic drag coefficient λ_m on parameter λ_H, computed for different values of Re, $\bar{\omega}$, q, and slip, is shown in Figs. 4-6. For small values of $\bar{\omega}$ and q (Fig. 4) this dependence is practically the same as in the case of constant field [6]. On applying the field the drag coefficient first decreases, attains a minimum, and then increases

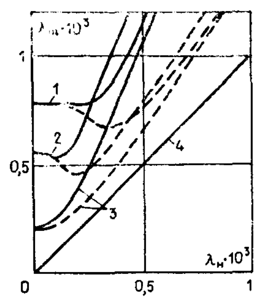

Fig. 6

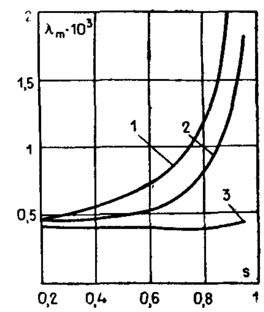

Fig. 7

Fig. 6. Dependence of the hydraulic drag coefficient λ_m on parameter λ_H for $\bar{\omega} = 2$, $q = 0.1$; 1) $Re = 3 \cdot 10^3$; 2) $Re = 10^4$; 3) laminar flow; 4) graph of $\lambda_m = \lambda_H$; continuous curves $s = 0.7$; dashes $s = 0.3$.

Fig. 7. Dependence of hydraulic drag coefficient λ_m on slip for $Re = 5000$, $\lambda_H = 5 \cdot 10^{-3}$; $\bar{\omega}$ and q: 1) 2 and 0.1; 2) 0.01 and 0.5; 3) 0.01 and 0.1.

again. The curves $\lambda_m = \lambda_m(\lambda_H)$, constructed for the same values of Re but different values of the slip (s = 0.3-0.7), merge into a single curve.

At large values of λ_H all the curves have a common asymptote close to the straight line $\lambda_m = \lambda_H$.

At large $\bar{\omega}$ and q the curves $\lambda_m = \lambda_m(\lambda_H)$, constructed for different s, coincide only on the initial segment (Figs. 5 and 6). The curves corresponding to larger values of the slip separate out from the common segment of the graph first. The graphs $\lambda_m = \lambda_m(\lambda_H)$ for laminar flow in traveling field (Figs. 5 and 6) behave in the same way in relation to s. For the computation for the laminar mode we put $\bar{l} = 0$ in formula (8). The curves $\lambda_m = \lambda_m(\lambda_H)$ for the turbulent flow, constructed for different values of Re, asymptotically approach the curves for the laminar flow for the same values of s, $\bar{\omega}$, and q as λ_H is increased. The hydraulic drag coefficient is always larger than λ_H and increases with s.

The dependence on the slip is shown in Fig. 7. At small values of $\bar{\omega}$, q the coefficient of friction is practically independent of the slip, which agrees with Harris' results [1]. When $\bar{\omega}$ and q are not small, the hydraulic drag coefficient increases with s; even in this case there is an appreciable range of slips $s \leq 0.5$, where λ_m remains practically unchanged.

4. Principal Results. The velocity profiles and the dependence of the hydraulic drag coefficient on the dimensionless parameters are computed for a turbulent flow in a traveling field; included among the dimensionless parameters is parameter λ_H, which depends on the magnetic field induction and is equal to the drag coefficient of laminar flow in a constant field with induction equal to the effective value of the induction of the total magnetic field in the investigated flow.

When parameter λ_H is used as the argument, the hydraulic drag coefficient remains practically independent of the slip in an appreciable range of s, which is in agreement with the results of Harris [1].

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