MOTION OF A CONDUCTING DROP IN AN INHOMOGENEOUS MAGNETIC FIELD

A. B. Isers and A. I. Él'kin UDC 537.84

1. In view of its applications [1, 2] it is of interest to investigated the flight of a drop of inviscous incompressible conducting liquid in a zone occupied by an inhomogeneous stationary magnetic field. We shall restrict ourselves to the case of small deformations of the drop in two-dimensional formulation of the problem. In the particular case of small deformations of the drop, we shall consider both two-dimensional and three-dimensional models. This comparison permits, to some extent, to get an idea of the feasibility of applying the results obtained on the basis of the two-dimensional model.

2. We consider the motion of a drop of inviscous incompressible conducting liquid in an inhomogeneous stationary magnetic field and in the field of gravity forces in vacuum in non-induction approximation.

In coordinate system K, moving with velocity **u**, the displacement and deformation of the drop are described by the following equations:

$$\operatorname{div} \mathbf{v}=0, \quad \mathbf{r} \in \Omega; \tag{1}$$

$$\rho \frac{d\mathbf{v}}{dt}=-\nabla p+\rho\left(\mathbf{g}-\frac{d\mathbf{u}}{dt}\right)+\mathbf{j}\times\mathbf{B}; \quad \mathbf{r}\in\Omega; \tag{2}$$

$$\mathbf{j}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B}), \quad \mathbf{r}\in\Omega; \tag{3}$$

$$\operatorname{rot} \mathbf{E}=\operatorname{rot}(\mathbf{u}\times\mathbf{B}), \quad \mathbf{r}\in\Omega; \tag{4}$$

$$\operatorname{div} \mathbf{j}=0; \quad \mathbf{j}\cdot\mathbf{n}=0, \quad \mathbf{r}\in\Gamma; \tag{5), (6}$$

$$\mathbf{v}\cdot\mathbf{n}=\frac{\partial\mathbf{r}}{\partial t}\cdot\mathbf{n}, \quad \mathbf{r}\in\Gamma; \quad p=2\varkappa\alpha, \quad \mathbf{r}\in\Gamma. \tag{7), (8}$$

Here **r** is the radius vector from the origin of K; Ω, region occupied by the drop at a given instant of time t; Γ, surface of the drop; **n**, unit vector of outward normal to Γ; 2ϰ, twice the mean curvature of Γ; α, surface tension coefficient; the rest of the notations are universal.

Equations (1)-(8) should be supplemented by the equation of motion of system K in a fixed system of coordinates K_1:

$$d\mathbf{R}/dt=\mathbf{u} \tag{9}$$

and the initial conditions. We take that at the initial instant of time t = 0 a) the drop was spherical and moved progressively; b) the initial velocity of the drop $\mathbf{u}_0=\mathbf{u}(0)$; c) system K coincided with system K_1. Then for t = 0

$$\mathbf{R}=0, \quad |\mathbf{r}|=l_*=\text{const}, \quad \mathbf{r}\in\Gamma; \quad \mathbf{v}=0, \quad \mathbf{r}\in\Omega. \tag{10}$$

Here **R** = (X,Y) is the radius vector of the origin of K in K_1; l_* is the initial radius of the drop.

Let vectors **u**, **g**, and **B** lie in a single plane and we assume that **B** changes only along this plain. Choosing the cartesian system of coordinates x, y, z in K such that **B** = (B_x(x, y), B_y(x,y), 0), and assuming that

Translated from Magnitnaya Gidrodinamika, No. 2, pp. 73-78, April-June, 1984. Original article submitted March 4, 1983.

 0024-998X/84/2002-0166$08.50

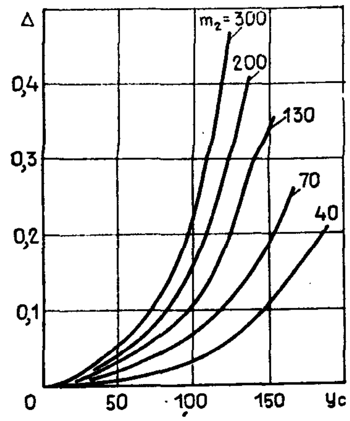

Fig. 1. Dependence of the ratio of the increase in velocity of the center of mass of the drop to its initial value on the length of flight y_c.

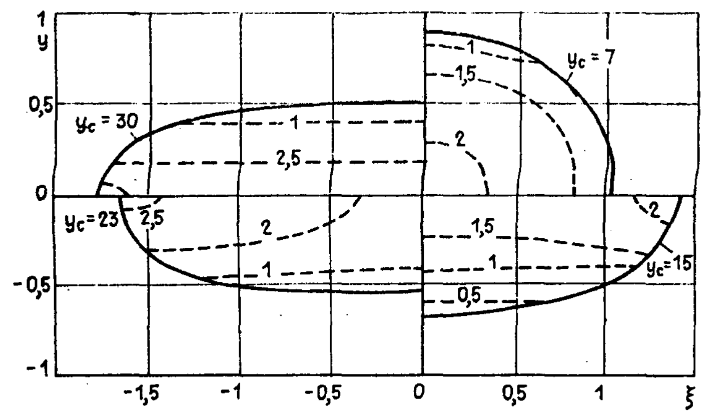

Fig. 2. Change of contour of drop during its motion.

$$\mathbf{v}=(v_x(x,y), v_y(x,y), 0), \quad p=p(x,y); \tag{11}$$

$$\mathbf{E}=(0,0,E(x,y)), \quad \mathbf{j}=(0,0,j(x,y)), \tag{12}$$

we obtain the simplified two-dimensional formulation of the problem.

From Eq. (4) we find that

$$\mathbf{E}=\mathbf{u}\times\mathbf{B}+\nabla\Phi,$$

where in view of (12) function Φ must have the form

$$\Phi=c(t)z+c_1(t),$$

from which we get

$$\mathbf{E}=\mathbf{u}\times\mathbf{B}+c(t)\mathbf{e}_z.$$

From the condition of conservation of current

$$\int_\Omega j\,dx\,dy=0,$$

which replaces (6) for the z component of the current density, and from Eq. (3) we get

$$c(t)=-\frac{1}{\pi l_*^2}\int_\Omega(\mathbf{u}+\mathbf{v})\times\mathbf{B}\,dx\,dy.$$

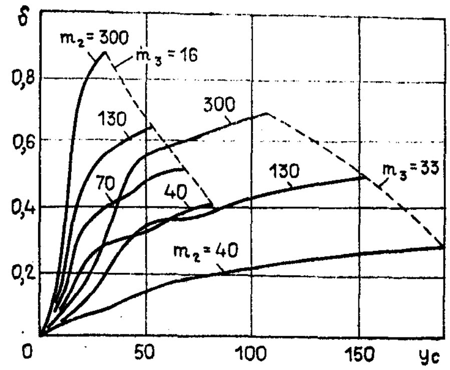

Fig. 3. Dependence of maximum deformation of the drop δ on the length of flight y_c,

Passing on to nondimensional variables and retaining the same notations for the nondimensional quantities as for the dimensional ones, under condition (11) we obtain the two-dimensional formulation of the problem:

$$\operatorname{div} \mathbf{v}=0, \quad (x, y)\in\Omega;$$

$$\frac{d\mathbf{v}}{dt}=\nabla p+m_1\mathbf{e}_y-m_3\frac{d\mathbf{u}}{dt}+m_2\mathbf{f}_m, \quad (x, y)\in\Omega; \tag{13}$$

$$\mathbf{f}_m=\left[\left(\mathbf{u}+\frac{\mathbf{v}}{m_3}\right)\times\mathbf{B}-\frac{1}{\pi}\int_\Omega\left(\mathbf{u}+\frac{\mathbf{v}}{m_3}\right)\times\mathbf{B}\,dx\,dy\right]\cdot\mathbf{e}_z(B_x\mathbf{e}_y-B_y\mathbf{e}_x),$$

$$\mathbf{v}\cdot\mathbf{n}=\frac{\partial}{\partial t}(x\mathbf{e}_x+y\mathbf{e}_y)\cdot\mathbf{n};$$

$$p=2x, \quad (x, y)\in\Gamma; \quad d\mathbf{R}/dt=m_3\mathbf{u}$$

and for t = 0

$$\mathbf{R}=0, \quad \mathbf{v}=0, \quad |\mathbf{r}|=1, \quad \mathbf{r}\in\Gamma. \tag{14}$$

Here the y axis is so chosen that $g = ge_y$. As the characteristic quantities we have used l_*, b_* (the characteristic gradient of the magnetic field induction), u_* (the characteristic velocity of system K),

$$p_*=\alpha/l_*, \quad v_*=\sqrt{p_*/\rho}, \quad t_*=l_*/v_*, \quad B_*=b_*u_*t_*, \quad E=u_*B_*, \quad j_*=\sigma E_*,$$
$$m_1=g\rho l_*^2/\alpha, \quad m_2=j_*B_*l_*^2/\alpha, \quad m_3=u_*/v_*.$$

3. A finite-difference method similar to one described in [3, 4] was used for the numerical solution of problem (13), (14). The computations were done in the real [1] range of variation of the parameters $m_1 = 0$, $m_2 = 40 - 300$, $m_3 = 16 - 50$ for $\bar{u} = u_0 e_y$,

$$\mathbf{B}=\begin{cases}(y/m_3+t)\mathbf{e}_x, & y/m_3+t\geqslant 0;\\ 0, & y/m_3+t<0,\end{cases}$$

which corresponds to the case of a current layer with homogeneous current density [5].

With this choice of u and B parameter m_3 specifies the nondimensional entry velocity of the drop, the deformation of the drop occurs symmetrically to the y axis, and the radius vector of the center of mass of the drop R_c and velocity of the center of mass v_c in system K_1 have only one nonzero component, $R_c = (0, y_c)$, $v_c = (0, v_c)$.

The results of computations are shown in Figs. 1-3.

Figure 1 shows the family of curves of dependence of the ratio Δ of the increase in the velocity of the center of mass of the drop to its initial value ($\Delta = (m_3 - v_c)/m_3$) on the length of flight y_c for $m_3 = 33$ and different values of m_2; m_2 characterizes the ratio of the pressure caused by the electromagnetic forces to the pressure due to surface tension forces. As seen from the figure, the motion of the drop is accompanied by its retardation.

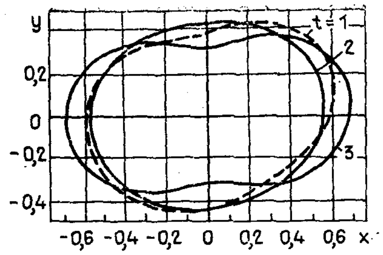

Fig. 4

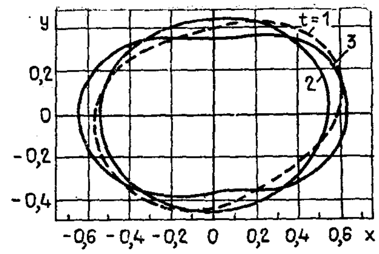

Fig. 5

Fig. 4. Change in contour of the drop in the case of small deformations for the two-dimensional model.

Fig. 5. Change in the contour of the drop in the case of small deformations in three-dimensional formulation.

The change in the contour of the drop during its motion is shown in Fig. 2 for m_2 = 200 and m_3 = 16. Computations showed that the asymmetry of Γ with respect to the ξ axis passing through the center of mass of the drop parallel to the x axis increases with the increase of m_2 and decrease of m_3. However, it remains insignificant in the entire range of variation of parameters m_2 and m_3. From graphics point of view the contour of the drop is symmetric to the ξ axis; therefore, for each value it is shown only in one quadrant. The isobars are plotted on the same figure and shown by the dashed lines.

During its motion the drop gets stretched in the direction of the magnetic field and becomes ellipsoidal in shape. Later as deformation progresses, its contour gets flattened in regions adjacent to the y axis after which the formation of neck occurs there. The region of increased pressure, which appears at the center of the drop at the start of the motion, gradually moves to its periphery along the ξ axis. A zone of increased pressure appears in regions adjacent to the ξ axis and zone of decreased pressue in regions of Γ adjacent to the y axis. Inside the drop the isobars are gradually stretched along the ξ axis and the pressure drop between the central and peripheral layers increases both due to the pressure increase in the central layers and pressure drop at the periphery.

The start of the disintegration of the drop was conventionally identified with the instant of development of the neck. Since the pressure is lower in the zone of the neck, a hastening of disintegration may be expected due to development of cavitation.

The family of curves of the dependence of maximum deformation $\delta=\max_{x,y\in\Gamma}\sqrt{(x-x_c)^2+y^2-1}$ in Γ on y_c is shown in Fig. 3 with m_2 and m_3 as parameters. Here x_c is the coordinate of the center of mass of the drop in system K. The dashed lines in the figure indicate the dependence of the length of flight of the drop and its deformation up to the instant of disintegration on the controlling parameters m_2 and m_3. We note that an increase of m_2 leads to a decrease of the length of flight and increase of deformation.

4. It is of interest to estimate the applicability of the two-dimensional approximation. For this purpose we consider the fall of a drop that is initially at rest with $m_2 \ll 1$ in a field of the form

$$\mathbf{B}=b[(y_c+y)\mathbf{e}_x+(x_c+x)\mathbf{e}_y]+b_x\mathbf{e}_x+b_y\mathbf{e}_y, \quad (15)$$

where b, b_x, b_y are constants; b = 1 in view of the choice of b_* and is retained for clarity. Here and below the origin of coordinate system K is taken at the center of mass of the drop, $u_* = gt_*$, $m_3 = m_1$.

In this case the problem can be solved analytically by the method of small parameter [6] both in two-dimensional and complete formulation.

The solution of problem (11), (13), (14), (15) has the form

$$r=1+m_2r^{(1)}+O(m_2^2); \quad (16)$$

$$r^{(1)}=a_2\cos 2\varphi+b_2\sin 2\varphi+b_3\sin 3\varphi, \quad \mathbf{r}\in\Gamma;$$
$$x_c=O(m_2^2), \quad y_c=m_1t^2/2-m_2t^3b^2/24+O(m_2^2). \quad (17)$$

Here (r, φ) is the polar system of coordinates in system K,

$$a_2=\frac{m_1b^2}{24}\left(t^3-t+\frac{\sin t\sqrt{6}}{\sqrt{6}}\right)+\left(\frac{bb_x}{12}\right)\left(t-\frac{\sin t\sqrt{6}}{\sqrt{6}}\right);$$

$$h=\frac{bb_y}{12}\left(t-\frac{\sin t\sqrt{6}}{\sqrt{6}}\right);\quad b_3=\frac{b^2}{48}\left(t-\frac{\sin 2t\sqrt{6}}{2\sqrt{6}}\right). \tag{18}$$

The solution of the problem, obtained with an accuracy up to $O(m_2^2)$, is not equally suitable for infinite time interval, since in (16) and (17) the coefficients of m_2 increase indefinitely with increase of t.

Formulas (16), (17) show that in the zero order approximation in m_2 the motion of the drop represents its free fall under the force of gravity. In view of the assumption of nonviscous liquid in the drop it does not get deformed. In the first approximation also the motion of the drop occurs along the direction of the force of gravity. The correction term of the order $O(m_2)$ in the velocity of the center of mass of the drop is negative, i.e., the drop gets retarded due to the interaction of the currents induced in it with the external magnetic field. Along with the gradient of the magnetic induction here the magnitude of B is also significant; however, this is true only when the gradient is nonzero. The surface of the drop executes an oscillatory motion about the y axis and tends to a symmetric form with time. The dependence of $r^{(1)}$ on φ is shown in Fig. 4 for $m_1 = 1.5$, $b_x = b_y = 1$; the dashed line corresponds to the function $0.5 + r^{(1)}$ at t = 1 and the continuous lines to the function $0.5 + 0.1\, r^{(1)}$ at t = 2 and t = 3.

In the case of the complete formulation, system (1)-(10) is brought to the nondimensional form with the same characteristic quantities as in (13), (14). The problem was written out in spherical coordinates and the solution was again found by the method of small parameter.

In this case the motion of the center of mass of the drop is described by the formulas:

$$x_c=O(m_2^2);\quad y_c=O(m_2^2);\quad y=m_1t^2/2-m_2t^3b^2/60+O(m_2^2). \tag{19}$$

In order to be able to compare the obtained solution with the solution in the two-dimensional formulation we also give the form S of the contour of section Σ of the drop by the plane z = 0:

$$\beta=1+m_2\beta^{(1)}+O(m_2^2);$$

$$\beta^{(1)}=A_0+B_2\sin 2\alpha+A_2\cos 2\alpha+B_3\sin 3\alpha. \tag{20}$$

Here (β, α) is the polar system of coordinates in xOy plane with origin at the center of mass of section Σ;

$$A_0=\frac{m_1b^2}{192}\left(t^3-\frac{3t}{4}+\frac{3}{4}\,\frac{\sin t\sqrt{8}}{\sqrt{8}}\right)+\frac{bb_x}{96}\cdot\left(t-\frac{\sin t\sqrt{8}}{\sqrt{8}}\right);\quad A_2=3A_0;$$

$$B_2=\frac{bb_y}{120}\left(t-\frac{\sin t\sqrt{8}}{\sqrt{8}}\right);\quad B_3=\frac{b^2}{120}\left(t-\frac{\sin t\sqrt{30}}{\sqrt{30}}\right). \tag{21}$$

A comparison of formulas (16)-(18) and (19)-(21) shows that for both the models the departure of the contour from a circle is described by the same harmonics in the polar angle with similar dependence of their amplitudes on the quantities controlling the motion. Therefore, it may be expected that the investigation of the deformation of the drop in the two-dimensional formulation offers the possibility of infering the tendencies in the deformation of the section of the drop by the plane of vectors **B** and **g**.

The dependence of $\beta^{(1)}$ on α is shown in Fig. 5 for $m_1 = 1.5$, $b_x = b_y = 1$; the dashed line corresponds to the function $0.5 + 2\beta^{(1)}$ at t = 1, the continuous lines to function $0.5 + 0.2\beta^{(1)}$ at t = 2 and t = 3.

A comparison of Figs. 5 and 4 substantiates the correctness of the assumptions made above.

We note that, as follows from formulas (16)-(18) and (19)-(21) and Figs. 4 and 5, the solution of the problem in two-dimensional formulation gives overestimated values of the degree of deformation and retardation of the real drop. Therefore the deformation of the drop, computed from the two-dimensional model, should apparently be taken as the upper estimate of the actual deformations during the motion of a conducting drop in an inhomogeneous stationary magnetic field.

LITERATURE CITED

1. E. V. Murav'ev, "Motion of conducting spherical particles in inhomogeneous magnetic field," Magn. Gidrodin., No. 3, 43-50 (1980).
2. E. V. Murav'ev, "Contact devices of diverter and limiter systems of tokamak reactors," Problems of Atomic Science and Technology, Termoyadernyi Sintez, No. 2 (6), 57-64 (1980).
3. C. W. Hirt, I. L. Cook, and T. D. Butler, "A Lagrangian method for calculating the dynamics of an incompressible fluid with free surface," J. Computational Phys., No. 5, 103-124 (1970).
4. N. N. Anuchina, K. I. Babenko, S. K. Godunov, N. A. Dmitriev, L. V. Dmitrieva, V. F. D'yachenko, A. V. Zabrodin, O. V. Lokutsievskii, E. V. Malinovskaya, I. F. Podlivaev, G. P. Prokopov, I. D. Safronov, and R. L. Fedorenko, Theoretical Bases and Construction of Numerical Algorithms for Problems of Mathematical Physics [in Russian], Nauka, Moscow (1979).
5. A. I. Vol'dek, Magnetohydrodynamic Induction Machines with Liquid Metal Operating Substance [in Russian], Énergiya, Leningrad (1970).
6. J. Cole, Perturbation Methods in Applied Mathematics [Russian translation], Mir, Moscow (1972).