STABILITY OF MHD JET FLOW BETWEEN NONCONDUCTING PLANES IN TRANSVERSE FIELD

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UDC 537.84

Introduction. The interest in the stability of jet-type flows within the scope of purely two-dimensional problem has grown considerably in recent years. This is due to the needs of modeling atmospheric phenomena [1] as well as many other technological processes [2]. MHD effects, on one hand, are used for modeling purely hydrodynamic processes such as Kolmogorov flow [3] and, on the other hand, are of independent interest.

The condition of small Hartmann numbers is typical for the problems of the first type, whereas the most pressing problems of the second type are those involving large Hartmann numbers.

In the present article we investigate the linear problem of stability of flow of a conducting fluid between nonconducting parallel planes normal to the induction vector of an external uniform magnetic field at large Hartmann numbers. This flow, for which the azimuthal velocities along z and r coordinates are shown in Fig. 1 (2 and 3; 1 is the axis of symmetry of the flow), is formed as a result of interaction of the magnetic field with the current supplied using two coaxial electrodes 4 and 5.

The theoretical investigation of an unperturbed flow of this type is presented in [4]. The results of an experimental investigation of the unperturbed rotating layer formed with the use of two coaxial electrodes are given in [5].

The stability of this type of flow has been investigated in [6, 7] by the energy method. It is shown that the increase in the stability of free shear flows is due to dissipation of energy of the perturbations in Hartmann layers. For thin electrodes the Reynolds number corresponding to the loss of stability is proportional to the square root of the Hartmann number, while for wide electrodes it is proportional to the Hartmann number. The coefficient of proportionality has been experimentally determined in [7].

Translated from Magnitnaya Gidrodinamika, No. 2, pp. 23-28, April-June, 1985. Original article submitted September 24, 1984.

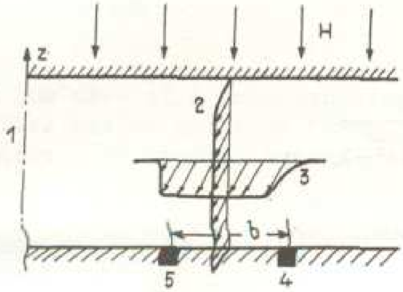

Fig. 1

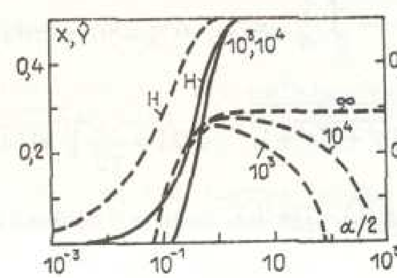

Fig. 2

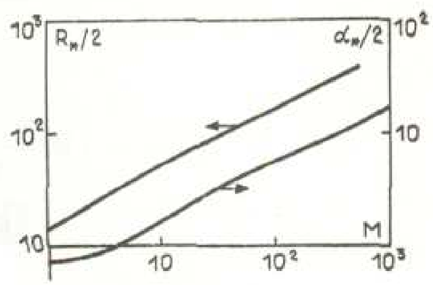

Fig. 3

Outside the Hartmann layers the unperturbed flow is nearly one-dimensional at large Hartmann numbers and the perturbations are nearly two-dimensional. In the investigation of stability a one-dimensional flow with two-dimensional perturbations may be treated by replacing the drag force in Hartmann layers by an equivalent force uniformly distributed along the coordinate parallel to the field. This method of approximate consideration of the effect of friction at the end walls is used in [1]. In [8, 9], in which the stability of MHD jet flows is investigated, the effect of Hartmann layers on the stability was not taken into consideration (in [8] the friction at the walls normal to the magnetic field was not discussed at all, while in [9] the effect of magnetic field on the friction is not taken into consideration).

In the present study (as also in [6, 7]) the critical effect of Hartmann layers on the stability is taken into consideration, but here it is modeled by the drag force.

We shall assume that the uniform magnetic field is applied parallel to the z axis in cylindrical coordinates (r, φ, z). We shall disregard the induced magnetic field, assuming, that the Reynolds number is much smaller than unity. Taking velocity component $v_z = 0$ and considering that an axial field has no effect on a flow which is transverse to it and uniform along z (except for the formation of Hartmann layers), we find that the components v_r, v_φ of the velocity perturbations satisfy the usual equations of hydrodynamics (for example, see [10]):

$$\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\varphi}{r}\frac{\partial v_r}{\partial \varphi}-\frac{v_\varphi^2}{r}+\frac{U}{r}\frac{\partial v_r}{\partial \varphi}-2\frac{Uv_\varphi}{r}=-\frac{\partial p}{\partial r}+\frac{1}{\mathrm{Re}}\left(\Delta_3 v_r-\frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi}\right);$$

$$\frac{\partial v_\varphi}{\partial t}+v_r\frac{\partial v_\varphi}{\partial r}+\frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial \varphi}+\frac{v_r v_\varphi}{r}+U'v_r+\frac{Uv_r}{r}=-\frac{1}{r}\frac{\partial p}{\partial \varphi}+\frac{1}{\mathrm{Re}}\left(\Delta_3 v_\varphi+\frac{2}{r^2}\frac{\partial v_r}{\partial \varphi}\right);\quad \frac{\partial r v_r}{\partial r}+\frac{\partial v_\varphi}{\partial \varphi}=0; \qquad (1)$$

$$\Delta_3 f\equiv\Delta_2 f+\frac{\partial^2 f}{\partial z^2},\quad \Delta_2 f=\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial r f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2}.$$

Here U(r) is the azimuthal (and the only) component of the velocity of the original flow. All variables in system (1) are dimensionless. The distance between the midpoints of the electrodes b (width of the velocity profile) is taken as the length scale, and the maximum value u_m serves as the velocity scale. Accordingly, the Reynolds number $\mathrm{Re} = u_m b/\nu$.

We carry out averaging operation $(1/h)\int_0^h dz$, where h is the dimensionless thickness of the layer. Near the lower boundary, for example, $v_r, v_\varphi \sim 1-\exp(-\mathrm{Ha}z)$. In Hartmann number Ha also b serves as the length scale. During averaging, all the terms in the Laplacian except the second derivatives with respect to z obtain a factor of the form $1 + O(1/\mathrm{Ha})$. Hereafter we shall neglect terms of the order of 1/Ha. The averaging of the second derivative of v over one of the Hartmann boundary layers gives the term $(-\mathrm{Ha}/h)v$; therefore, when both layers are considered, the operator $\partial^2/\partial z$ goes over into $-M$, where M is the modified Hartmann number $M \equiv 2\mathrm{Ha}/h$. After averaging, the equations for the perturbations acquire the previous form if the same notations are used for the averaged quantities but operator Δ_3 is given a new meaning:

$$\Delta_3 f \to \Delta_2 f - Mf.$$

Thus, we obtain a two-dimensional hydrodynamic problem for the perturbations in which the effect of the magnetic field reduces to the appearance of an effective frictional force, i.e., to terms $(-M/\mathrm{Re})v$ in the right-hand side of (1).

Plane Jet. Here we shall consider the case when the width of the jet is much smaller than the radius of curvature and, therefore, the initial flow can be regarded as plane-parallel. In this case the linear stability problem reduces to the classical Orr–Sommerfeld equation [10]:

$$\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi=i\alpha\,\mathrm{Re}[(U-\hat{C})(\varphi''-\alpha^2\varphi)-U''\varphi].$$

Here we have used the notation $\hat{C} = C + iM/\alpha\mathrm{Re}$; $C = X + iY$ is the complex phase velocity of the perturbation, $\varphi(y)$ is the complex amplitude of the stream function, and wave number $\alpha = m/r_0$, where m is the azimuthal number, r_0 is the characteristic radius of curvature, and α is finite for the limiting transition r_0, $m \to \infty$.

Computational and experimental results [5] indicate that at large Hartmann numbers the asymptotic velocity profile of the jet may be approximated by the trapezoid

$$U=\begin{cases} 1\,; & |y|<(1-\Delta)/2,\\ (1+\Delta-2|y|)/(2\Delta); & 1-\Delta\leqslant 2|y|\leqslant 1+\Delta,\\ 0\,; & |y|>(1+\Delta)/2. \end{cases} \tag{2}$$

Here Δ is the dimensionless width of the electrodes.

The quantities C and $\hat{C}$ differ only in their imaginary parts: $\hat{Y} = Y + M/\alpha\,\mathrm{Re}$. The neutral curve $Y = 0$ for M = const corresponds to the level line $\alpha\,\mathrm{Re}\,\hat{Y}$ = const > 0. First we consider the case of infinitely thin electrodes, $\Delta = 0$. In this case the velocity profile becomes discontinuous: $U = 1$, $|y| < 0.5$ and $U = 0$, $|y| > 0.5$.

Rayleigh Problem for Infinitely Thin Electrodes ($\Delta = 0$). Since the asymptotic form $M \gg 1$ is being investigated, it is natural to consider the limiting case $\mathrm{Re} \to \infty$, Re/M finite. The Rayleigh equation corresponding to this case is

$$(U-\hat{C})(\varphi''-\alpha^2\varphi)=U''\varphi.$$

Since here U" is derivatives of Dirac delta function and so is φ'', the above equation can be more conveniently written in the form

$$[(U-\hat{C})\varphi'-U'\varphi]'=\alpha^2(U-\hat{C})\varphi.$$

Since the right-hand side has discontinuity of no more than the first kind, the expression in the square brackets is continuous: $(U-\hat{C})\varphi'-U'\varphi=\chi$, where χ is a continuous function. Dividing both sides by $(U-\hat{C})\varphi$ we find that $[\ln\varphi/(U-\hat{C})]'$ has only a discontinuity of the first kind and, therefore, $\varphi/(U-\hat{C})$ is continuous. Denoting the solution for $|y| < 0.5$ by φ_+ and that for $y > 0.5$ by φ_-, we obtain the coupling conditions at the point of discontinuity $y = 0.5$:

$$(1-\hat{C})\varphi_+'=-\hat{C}\varphi_-'; \quad (1-\hat{C})\varphi_-=-\hat{C}\varphi_+. \tag{3}$$

Since $\varphi''-\alpha^2\varphi=0$ inside each region, satisfying the requirements of damping at $y \to \infty$ and symmetry at $y = 0$, we have $\varphi_+ = A\cosh\alpha y$; $\varphi_- = B\exp(-\alpha y)$ and by virtue of coupling conditions (3) we get the dispersion equation

$$(1-\hat{C})^2\,\mathrm{th}\,\alpha_1+\hat{C}^2=0; \quad \alpha_1=\frac{\alpha}{2}; \quad X=\frac{\mathrm{th}\,\alpha_1}{1+\mathrm{th}\,\alpha_1}, \quad \hat{Y}=\frac{\sqrt{\mathrm{th}\,\alpha_1}}{1+\mathrm{th}\,\alpha_1}. \tag{4}$$

We note that if instead of the symmetry condition we set antisymmetry condition, then $\tanh\alpha_1$ is replaced by $\coth\alpha_1$. Here $\hat{Y}$ remains invariant and the phase velocity becomes equal to $X = 1/(1 + \tanh\alpha_1)$. Thus, symmetric perturbations propagate with velocity $0 < X < 0.5$, while antisymmetric ones propagate with velocity $0.5 < X < 1$.

The equation of the neutral curve is $\mathrm{Re} = M(\sqrt{\tanh\alpha_1} + \sqrt{\coth\alpha_1})/\alpha$. The Reynolds number tends to zero as α increases, i.e., this analysis gives unstable flow at all Reynolds numbers.

However, since the critical point, where the phase velocity and jet velocity coincide, occurs exactly at the point of discontinuity of the U profile, the viscosity occurring as the coefficient of the first derivatives may have an effect even in the limit $\mathrm{Re} \to \infty$. As shown below this is exactly what happens.

Orr–Sommerfeld Problem for $\Delta = 0$. For M = 0 this problem has been investigated in [11]. The conditions at the point of discontinuity of U are still less stringent, φ and φ' are continuous, and the conditions for the first derivatives are

$$\varphi_+''-\varphi_-''=-i\alpha\,\mathrm{Re}\,\varphi; \quad \varphi_+'''-\varphi_-'''=i\alpha\,\mathrm{Re}\,\varphi'.$$

As before the dispersion equation is derived analytically, but it is unwieldy (see [11]), and for determining C the roots of transcendental equation must be determined and numerical computations have to be used.†

The results of the computations are shown in Fig. 2. The real part of $\hat{C}$ as a function of α is shown by the solid curves, while the imaginary part is shown by the dashed curves. The results of the previous section [formula (4)] are marked by letter H. Near the curves corresponding to finite viscosity the half values of Re, for which they are computed, are shown.

For the phase velocity the curves for $Re/2 = 10^3$ and 10^4 are indistinguishable within the accuracy of the graphical representation. For sufficiently large values of α, $X \to 0.5$ for $\alpha \to \infty$ in all cases. On the contrary, the divergence between the values of $\hat{Y}$ remains finite for all values of α and in the limit $Re \to \infty$, although the qualitative behavior, i.e., approaching a constant value for $\alpha \to \infty$, remains the same. In the Rayleigh problem the limit value is $\hat{Y} = 0.5$, and for the limiting transition $Re \to \infty$ $\hat{Y} \to 0.286$. The larger difference in the imaginary part is due to the fact that the viscous term in the Orr–Sommerfeld equation contain imaginary unit.

In the theory of hydrodynamic stability the Lin alternative [12] is well known; it states that the result of the Rayleigh problem coincides with the limit for $Re \to \infty$ when instability occurs ($Y > 0$), while in the opposite case ($Y < 0$) generally there is no such agreement. The solutions presented here indicate that for discontinuous velocity profiles a regular limiting transition may not occur even in the case of positive Y.

As already mentioned, for fixed induction the equation of the neutral curves is of the form $M = \alpha \, Re \, \hat{Y} = \text{const}$. The dependence of the critical parameters Re_*, α_* (corresponding to the minimum Re on the neutral curves) on M is shown in Fig. 3. The results for $M \gg 1$ are of maximum interest. In this case the following asymptotic relations are obtained:

$$Re_* \approx 34\sqrt{M}; \quad \alpha_* \approx 1.09\sqrt{M}; \quad X_* = 0.5. \tag{5}$$

They agree with the results of [7]. The asymptotic neutral curve is shown in Fig. 4 in variables $\hat{\alpha} = \alpha/2\sqrt{M}$ and $\hat{R} = Re/2\sqrt{M}$. For $\hat{R} > 10^2$, $\hat{R} \sim \hat{\alpha}$ on the upper branch, while $\hat{R} \sim 1/\hat{\alpha}$ on the lower branch.

Rayleigh Problem for $\Delta \neq 0$. As shown in [7], for a finite width of the electrodes the nature of the asymptotic dependence $Re_*(M)$ is now different from the case $\Delta = 0$. For analyzing the case $\Delta \neq 0$ we again return to the Rayleigh equation. Since the velocity profile is continuous now, one should expect a better agreement between the results obtained by the limiting transition $Re \to \infty$ directly in the equation or only in the solutions. Actually, in the region of positive decrements this agreement has been confirmed for jet flows by direct computations (for example, see [10]). Here the breaks in the velocity profile are not a controlling factor, firstly, because the convergence of the Rayleigh method has been proved [13] and, secondly, because now the critical point for the neutral oscillations does not coincide with the break points, which dilutes the effect of viscosity.

We divide the region of flow in three subregions in accordance with (2). Considering the symmetry condition we shall seek the solution in subregion $|y| \leqslant y_1 \equiv (1 - \Delta)/2$ in the form $\varphi_0 = A\cosh \alpha y$, in subregion $y_1 < y < y_2 \equiv (1 + \Delta)/2$ in the form $\varphi_1 = B\exp \alpha y + D\exp(-\alpha y)$, and, finally, for $y > y_2$ in the form $\varphi_2 = E\exp(-\alpha y)$ in conformity with the damping requirement. The coupling conditions at the break points [continuity conditions and first of conditions (3)] give

$$(1-C)\varphi_0' - (1-\hat{C})\varphi_1' + U'\varphi_1 = 0; \quad \varphi_0 = \varphi_1; \quad y = y_1;$$

$$-\hat{C}\varphi_1' - U'\varphi_1 + \hat{C}\varphi_2' = 0; \quad \varphi_1 = \varphi_2; \quad y = y_2,$$

from which we get the dispersion equation

$$\begin{vmatrix} \alpha \operatorname{sh} \alpha y_1 & [U'/(1-\hat{C}) - \alpha] \exp \alpha y_1 & [U'/(1-\hat{C}) + \alpha] \exp(-\alpha y_1) & 0 \\ \operatorname{ch} \alpha y_1 & -\exp \alpha y_1 & -\exp(-\alpha y_1) & 0 \\ 0 & [U'/\hat{C} + \alpha] \exp \alpha y_2 & [U'/\hat{C} - \alpha] \exp(-\alpha y_2) & \alpha \exp(-\alpha y_2) \\ 0 & -\exp \alpha y_2 & -\exp(-\alpha y_2) & \exp(-\alpha y_2) \end{vmatrix} = 0.$$

†The authors are grateful to E. P. Kurochkinova for help in computations for the results of this section.

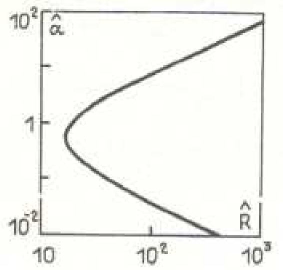

Fig. 4

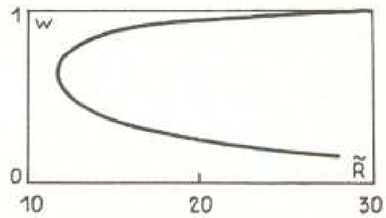

Fig. 5

After a number of manipulations and considering that $U' = -1/\Delta$ we get

$$2\hat{C}(1+\text{th}\,W)^2(1+\text{th}\,\alpha y_1) = (1+\text{th}\,W)^2(1+\text{th}\,\alpha y_1) - \text{th}\,W(1-\text{th}\,\alpha y_1)/W$$

$$\pm i\{(1+\text{th}\,\alpha y_1)^2\,(1+\text{th}\,W)^2[(1+\text{th}\,W)^2(2-W)/W - 2\,\text{th}\,W(1+W)/W^2] - [\text{th}\,W(1-\text{th}\,\alpha y_1)/W]^2\}^{1/2}; \quad W = \alpha\Delta/2.$$

In the case of small Δ and large M the critical value $\alpha_* \gg 1$, as follows, in particular, from the preceding analysis. Therefore, we put $\tanh \alpha y_1 = 1$. Thus, it at once follows that $X = 0.5$ and the equation of the neutral curve in variables $\tilde{R}$, W has the form

$$\tilde{R} \equiv \frac{4R}{M\Delta} = \frac{4(1+\text{th}\,W)}{W[(1+\text{th}\,W)^2(2-W)/W - 2\,\text{th}\,W(1+W)/W^2]^{1/2}}.$$

This curve is plotted in Fig. 5. The upper branch asymptotically tends to the straight line $W = W_m \approx 1.031$ and $\tilde{R} \approx 5.2\sqrt{W_m - W}$. Along the lower branch $W \to 0$ for $\tilde{R} \to \infty$ in accordance with the approximate dependence $\tilde{R} \approx (1 + W)/W$. For not too small values of Δ stagnation primarily remains at the lower branches, whereas the upper branch will be common.

The critical values corresponding to the obtained neutral curve are $\tilde{R}_* = 11.86$, $W_* = 0.64$. In the notations of [7] ($M = 4M^*$, $f = 2\Delta$) this corresponds to the asymptotic dependence

$$\text{Re}_* \approx 5.9M^* f, \quad \alpha_* \approx 1.28/\Delta. \tag{6}$$

Formula (6) is in qualitative agreement with expressions (4)-(5) of [7], but here the coefficient in the dependence for the critical Reynolds number is considerably (approximately by a factor of 5) smaller. This may be, in particular, due to the fact that in our idealization of the velocity profile there are breaks, whereas in real cases the discontinuities are smoothed out to scales $\sim 1/\sqrt{M}$ (the shape of the profile has a significant effect on the stability). Furthermore, in the experiment the radius of curvature is not too large ($r_0 = 3.5$), while here we have considered the limit $r_0 \to \infty$. Finally, the modeling of Hartmann layers by the drag force introduces its own error.

Thus, the obtained asymptotic dependence of the Reynolds number corresponding to the loss of stability on the modified Hartmann number agrees with the results of energy analysis [6, 7]. However, the theoretical numerical factor was found to be considerably smaller than the experimental [7], which is apparently related to the simplified construction of the theoretical model.

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