STABILITY OF FLOW OF A CONDUCTING LIQUID FILM IN AN INCLINED MAGNETIC FIELD

V. A. Bernshtam, S. V. Kozyrev,
E. V. Neznamova, and A. I. Él'kin

UDC 537.84

The investigation of the effect of electric and magnetic fields of various configurations on the stability of flow of a conducting liquid film is of current interest [1]. In [2-4] the stability of the film flow was investigated in the cases when the magnetic field has no effect on the profile of the basic flow. In [5, 6] the effect of changing velocity profile in a magnetic field perpendicular to the underlying surface was taken into consideration. In [7] the stability was investigated for small magnetic Reynolds numbers ($Re_m \precsim 1$) and small Hartmann numbers in a magnetic field having arbitrary orientation in the plane of the flow.

In the present work we investigate the stability of the laminar flow of an electrically conducting liquid film in crossed electric and magnetic fields for arbitrary Hartmann numbers in noninduction approximation. It is assumed that the magnetic induction vector lies in the plane of the flow and forms an arbitrary angle β to the underlying surface which is at angle θ to the horizon (Fig. 1), and the electric field vector E_z is perpendicular to the magnetic induction vector.

With these assumptions the flow under investigation is described by the following system of equations written in dimensionless form:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\sin\theta}{\mathrm{Fr}^2}-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\Delta u-K\frac{\mathrm{Ha}^2}{\mathrm{Re}}\sin\beta-u\frac{\mathrm{Ha}^2}{\mathrm{Re}}\sin^2\beta+v\frac{\mathrm{Ha}^2}{\mathrm{Re}}\sin\beta\cos\beta, \quad (1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{\cos\theta}{\mathrm{Fr}^2}-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\Delta v+K\frac{\mathrm{Ha}^2}{\mathrm{Re}}\cos\beta+u\frac{\mathrm{Ha}^2}{\mathrm{Re}}\sin\beta\cos\beta-v\frac{\mathrm{Ha}^2}{\mathrm{Re}}\cos^2\beta,$$

Translated from Magnitnaya Gidrodinamika, No. 2, pp. 42-46, April-June, 1985. Original article submitted October 23, 1984.

0024-998X/85/2102-0137$09.50

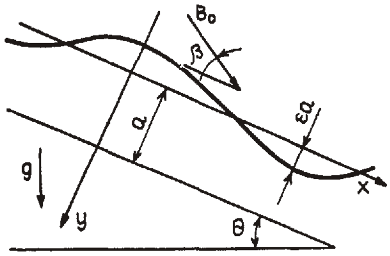

Fig. 1

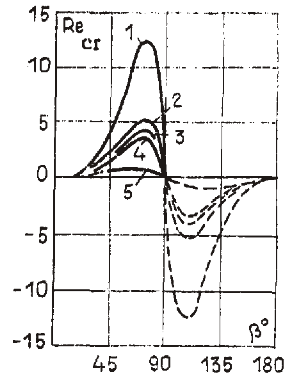

Fig. 2

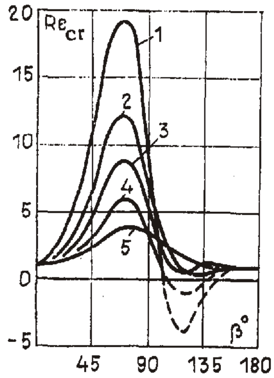

Fig. 3

Fig. 1. Geometry of the problem.

Fig. 2. Dependence of the critical Reynolds number on angle β. θ = 90°; K and Ha: 1) 1 and 1.5; 2) 0 and 1.3; 3) 1 and 1.3; 4) 2 and 1.3; 5) 1 and 1.

Fig. 3. Dependence of the critical Reynolds number on angle β. θ = 30°; K and Ha: 1) 1 and 1.5; 2) 0 and 1.3; 3) 1 and 1.3; 4) 2 and 1.3; 5) 1 and 1.

where Re and Ha are the Reynolds and Hartmann numbers, respectively.

System (1) must be supplemented by the equation of balance of the liquid in the film

$$\frac{\partial\varepsilon}{\partial t}+\frac{\partial}{\partial x}\int\limits_{\varepsilon}^{1} u(y)\,dy=0,$$

where ε is the coordinate of the free surface (Fig. 1), and boundary conditions similar to those used in [4]:

$$\begin{gathered}\text{I. } u(1)=0;\quad \text{II. } v(1)=0;\\ \text{III. } \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0,\quad y=\varepsilon;\\ \text{IV. } -p+\frac{2}{\mathrm{Re}}\frac{\partial v}{\partial y}+\frac{1}{\mathrm{We}}\frac{\partial^2\varepsilon}{\partial x^2}=0.\\ y=\varepsilon.\end{gathered}\tag{2}$$

Here $\mathrm{Re} = \rho\bar{U}a/\nu$, $\mathrm{Ha}^2 = B_0a^2\sigma/\nu$, $K = E_z/\bar{U}B_0$, $\mathrm{Fr}^2 = \bar{U}/ga$; B_0 is the modulus of the magnetic induction vector; a is the thickness of the film in plane-parallel flow; $\bar{U}$ is the mean velocity of the flow; $\mathrm{We} = \rho a\bar{U}/T$ is Weber number; ρ and σ are the density and conductivity of the liquid metal; ν is the kinematic viscosity; u, v are velocity components of the liquid.

For the unperturbed flow the velocity and pressure distributions over the thickness of the film and also the relationship between the defining parameters are of the form

$$U=\frac{\lambda}{\lambda-\operatorname{th}\lambda}\left(1-\frac{\operatorname{ch}\lambda y}{\operatorname{ch}\lambda}\right);\tag{3}$$

$$P=\left(\frac{\cos\theta}{\mathrm{Fr}}-\frac{\mathrm{Ha}^2}{\mathrm{Re}}K\cos\beta-\frac{\lambda\,\mathrm{Ha}\cos\beta}{(\lambda-\operatorname{th}\lambda)\mathrm{Re}}\right)y+\frac{\mathrm{Ha}\cos\beta}{\mathrm{Re}}\lambda\operatorname{th}\lambda y;\tag{4}$$

$$\mathrm{Fr}^2=\frac{\sin\theta\,\mathrm{Re}\,(\lambda-\operatorname{th}\lambda)}{\lambda^2(\lambda-(\lambda-\operatorname{th}\lambda)K/\sin\beta)};\tag{5}$$

$$\lambda=\mathrm{Ha}\sin\beta.\tag{6}$$

The load coefficient K occurring in Eqs. (1) determines whether the film is accelerated by the external current (K/sin β > 1) or flows between the dielectric walls (K/sin β = 1) or between the short-circuited conducting walls (K/sin β = 0).

The problem of stability of this flow within the premises of the linear theory of hydrodynamic stability reduces to the solution of the Orr—Sommerfeld equation [8]; for the present problem this equation is written in the form

$$\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi=i\alpha\,\mathrm{Re}\,((U-c)(\varphi''-\alpha^2\varphi)-U''\varphi)+\mathrm{Ha}^2(\varphi''\sin^2\beta+i\alpha\sin^2\beta\varphi'-\alpha^2\cos^2\beta\varphi), \quad (7)$$

where $\varphi(y)$ is the amplitude of the stream function of the velocity perturbations written in the form

$$\psi=\varphi(y)\exp i\alpha(x-ct),$$

where $\alpha = 2\pi a/\lambda_b$ is the dimensionless wavelength and $c = c_r + ic_i$ is the velocity of propagation of the perturbations, obtained as the eigenvalue of the problem. The imaginary part c_i determines the rate of growth ($c_i > 0$) or damping ($c_i < 0$) of the perturbations.

The boundary conditions for φ are of the form

$$\text{I. } \varphi(1)=0;\quad \text{II. } \varphi'(1)=0;\quad \text{III. } \varphi''(0)+\varphi(0)\left(\alpha^2+\frac{U''(0)}{c'}\right)=0;$$

$$\text{IV. } \varphi'''(0)-3\alpha^2\varphi(0)+\mathrm{Re}\,i\alpha\varphi(0)\left(-\frac{p'(0)}{c'}+U'(0)-\right.$$

$$\left.-\frac{\mathrm{Ha}^2\sin\beta\cos\beta}{\mathrm{Re}}\right)+i\alpha\varphi'(0)\,\mathrm{Re}\left(c'-\frac{1}{\mathrm{Re}}\mathrm{Ha}^2\sin\beta\right)+i\alpha^3\frac{\mathrm{Re}\,\varphi(0)}{\mathrm{We}\,c'}=0;\quad c'=c-U(0). \quad (8)$$

The dimensionless displacement ε is eliminated from the third and fourth boundary conditions with the use of the kinematic condition at the free surface, while number Fr is eliminated using (5).

Below we shall restrict the investigation to long-wave perturbations which are the most hazardous [8].

We seek the solution in the form of series in small parameter α by the method of successive approximations [8]. We put

$$\varphi=\varphi_0+\alpha\varphi_1+\alpha^2\varphi_2+\ldots;\quad c=c_0+\alpha c_1+\alpha^2c_2+\ldots$$

Unlike the cases investigated earlier here a term $\mathrm{Ha}^2 i\alpha\sin 2\beta\varphi'$ appears in the Orr–Sommerfeld equation, which takes account of the interaction of the current, produced by the motion of the conducting liquid in the transverse magnetic field, with the longitudinal component of the field. The same force is taken into consideration in boundary condition (8).

Since in the equation and the boundary conditions the terms taking account of the force $\mathbf{j}\times\mathbf{B}_0$ are proportional to α, the zero-order approximation coincides with the case of transverse magnetic field [6]:

$$c_0=\frac{\lambda\,\mathrm{th}^2\lambda}{\lambda-\mathrm{th}\,\lambda};\quad \varphi_0=\frac{2}{\lambda^2}(\mathrm{ch}\,\lambda(y-1)-1). \quad (9)$$

In the first approximation the system of equations and the boundary conditions will be of the form

$$\varphi_1^{IV}-\lambda^2\varphi_1=i\alpha\,\mathrm{Re}\,((U-c_0)\varphi_0''-U''\varphi_0)+i\alpha\,\mathrm{Ha}^2\sin 2\beta\varphi_0';$$

$$\varphi_1(1)=0;\quad \varphi_1'(1)=0; \quad (10)$$

$$\varphi_1'(0)+U''(0)\varphi_1(0)/c_0'-U''(0)\varphi_0(0)c_1/c_0'=0;\quad c_0'=c_0-U(0);$$

$$\varphi_1'''(0)-\lambda^2\varphi_1'(0)=-i\varphi_0(0)\left(\frac{dp}{dy}(0)\frac{\mathrm{Re}}{c_0'}-\lambda\,\mathrm{Ha}\cos\beta+i\,\mathrm{Re}\,c_0'\right). \quad (11)$$

We write the general solution of this equation in the form

$$\varphi_1(y)=\varphi^*+A_1\,\mathrm{ch}\,\lambda y+B_1\,\mathrm{sh}\,\lambda y+C_1y+D_1, \quad (12)$$

where

$$\varphi^*=y(A\,\mathrm{sh}\,\lambda y+B\,\mathrm{ch}\,\lambda y)/2\lambda^3.$$

The constant of integration is put equal to zero as in the zero-order approximation [8]. After substituting (9), (12) into (11) we get

$$A_1\,\mathrm{ch}\,\lambda+B_1\,\mathrm{sh}\,\lambda+C_1=A_2,\ A_1\,\mathrm{sh}\,\lambda+B_1\,\mathrm{ch}\,\lambda+C_1/\lambda=A_3,\ A_1=A_4(c_1),\quad C_1=A_5.$$

Here

$$A_2 = -(B\,\mathrm{ch}\,\lambda + A\,\mathrm{sh}\,\lambda)/2\lambda^3,$$

$$A_3 = -(B\,\mathrm{ch}\,\lambda + A\,\mathrm{sh}\,\lambda)/2\lambda^4 - (B\,\mathrm{sh}\,\lambda - A\,\mathrm{ch}\,\lambda)/2\lambda^3,$$

$$A_4 = A(\mathrm{ch}\,\lambda - 1)/\lambda^4 + c_1\, 2\,\mathrm{ch}^2\lambda\,(\lambda - \mathrm{th}\,\lambda)/\lambda^3,$$

$$A_5 = \frac{B}{\lambda^3} + 2i\alpha\left(\frac{(\lambda - \mathrm{th}\,\lambda)\,\mathrm{ch}^2\lambda}{\lambda^5} - \frac{\mathrm{Re}\cos\theta}{\mathrm{Fr}^2} + \right.$$

$$\left. + \mathrm{Ha}^2\cos\beta K\,\frac{(\lambda - \mathrm{th}\,\lambda)\,\mathrm{ch}^2\lambda}{\lambda^5} - \frac{\mathrm{Re}\,\mathrm{sh}\,\lambda\,(\mathrm{ch}\lambda - 1)}{\lambda^2(\lambda - \mathrm{th}\,\lambda)\,\mathrm{ch}^2\lambda} + \mathrm{Ha}\cos\beta\,\frac{(\mathrm{ch}\,\lambda - 1)^2}{\lambda^3}\right),$$

where

$$A = -i\alpha\,\mathrm{Ha}^2\sin 2\beta\,\frac{2\,\mathrm{sh}\,\lambda}{\lambda};$$

$$B = -2i\alpha\,\mathrm{Re}\,\frac{\lambda\,\mathrm{sh}\,\lambda}{(\lambda - \mathrm{th}\,\lambda)^2\,\mathrm{ch}^2\lambda} + i\alpha\,\mathrm{Ha}^2\sin 2\beta\,\frac{2\,\mathrm{ch}\,\lambda}{\lambda}.$$

Solving the system for c_1 we get

$$c_1 = ic_i = i\left(\frac{\mathrm{Re}\,(3\,\mathrm{sh}\,\lambda\,\mathrm{ch}^2\lambda\,(\lambda - \mathrm{th}\,\lambda) - \lambda\,\mathrm{sh}^3\lambda)}{2\,\mathrm{ch}^5\lambda\,(\lambda - \mathrm{th}\,\lambda)^2}\right.$$

$$\left. - \frac{\mathrm{ctg}\,\theta\,(\lambda - (\lambda - \mathrm{th}\,\lambda)K/\sin\beta)}{\lambda} + \frac{\mathrm{Ha}^2\cos\beta K(\lambda - \mathrm{th}\,\lambda)}{\lambda^3} - \frac{\mathrm{Ha}\cos\beta\,(\mathrm{ch}^2\lambda\,(\lambda - \mathrm{th}\,\lambda) + 2(\lambda - \mathrm{sh}\,\lambda))\,\mathrm{ch}\,\lambda}{\lambda(\lambda - \mathrm{th}\,\lambda)}\right).$$

From the neutral stability condition $c_i = 0$ we get the expression for the critical Reynolds number:

$$\mathrm{Re}_{cr} = \mathrm{ctg}\,\theta\,\frac{2\,\mathrm{ch}^2\lambda\,(\lambda - \mathrm{th}\,\lambda)^2(\lambda - (\lambda - \mathrm{th}\,\lambda)K/\sin\beta)}{\mathrm{th}\,\lambda\,(3(\lambda - \mathrm{th}\,\lambda) - \lambda\,\mathrm{th}^2\lambda)} - \frac{\mathrm{Ha}^2\cos\beta K 2(\lambda - \mathrm{th}\,\lambda)\,\mathrm{ch}^2\lambda}{\lambda^3\,\mathrm{th}\,\lambda\,(3(\lambda - \mathrm{th}\,\lambda) - \lambda\,\mathrm{th}^2\lambda)}$$

$$+ \mathrm{Ha}\cos\beta\,\frac{2(\lambda - \mathrm{th}\,\lambda)\,(\mathrm{ch}^2\lambda\,(\lambda - \mathrm{th}\,\lambda) + 2(\lambda - \mathrm{sh}\,\lambda))\,\mathrm{ch}^3\lambda}{\lambda\,\mathrm{th}\,\lambda\,(3(\lambda - \mathrm{th}\,\lambda) - \mathrm{th}^2\lambda)}. \tag{13}$$

This expression coincides with those obtained earlier in [6, 8, 9] for the limiting cases.

The first term in (13) describes the effect of the component of the magnetic induction vector perpendicular to the underlying surface and is similar to the expression obtained in [6]. The remaining terms describe the effect of the interaction of currents caused by the transverse component of the magnetic induction vector and by the electric field with longitudinal component.

The neutral stability curves, constructed from (13), are shown in Figs. 2-5.

Figure 2 shows that in contrast to the longitudinal or perpendicular field the inclined magnetic field can stabilize the film flowing along the vertical wall. The range of angle of inclination of the magnetic field induction vector to the underlying surface, for which stabilization is possible, is included between small acute angles and the angle $\beta = 90°$. The dependence $\mathrm{Re}_{cr} = \mathrm{Re}_{cr}(\beta)$ is odd with respect to $\beta = 90°$ and has its maximum at $\beta = 70$-$80°$. With the increase of load coefficient K critical Reynolds numbers decrease. The surface of the film can be stable for $K > 1$, when the total force acting on the unperturbed film tends to break it away from the underlying surface. This type of dependence can apparently be explained qualitatively by the interaction of a number of perturbed electric volume forces. These include the forces caused only by the component of the magnetic field perpendicular to the underlying surface [6], forces appearing due to the interaction of the perturbed current, caused by the electric field, with the longitudinal component of the induction, and finally the forces arising due to the interaction of longitudinal velocity perturbations with the transverse magnetic field.

The last force is proportional to $\sin 2\beta$ and predominates in the examples presented here; this qualitatively explains the nature of the curves in Fig. 2.

For the film flowing along the underlying surface placed at an angle to the horizon (Fig. 3) the region of stability over angle β becomes wider, since in this case stable flows

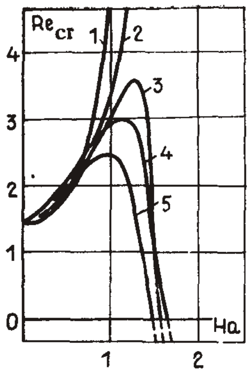

Fig. 4

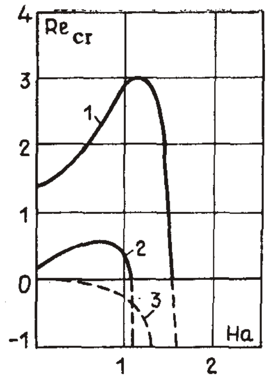

Fig. 5

Fig. 4. Dependence of the critical Reynolds number on the Hartmann number. θ = 30°, K and β: 1) 1 and 18°; 2) 1 and 90°; 3) 1 and 108°; 4) 1 and 99°; 5) 1.5 and 108°.

Fig. 5. Dependence of the critical Reynolds number on the Hartmann number. β = 99°; θ: 1) 30°; 2) 60°; 3) 90°.

are possible for β= 90°. With the increase of the Hartmann number (Fig. 4) for β < 90° the critical Reynolds number increases, whereas for β > 90° the function Re = Re(Ha) has a maximum and at sufficiently large Hartmann number the flow becomes unstable.

LITERATURE CITED

1. E. V. Murav'ev, "Contact properties of diverter and limiter systems of Tokamak reactors. 1. Devices with liquid metal operating surface," Problems of Atomic Science and Technology. Thermonuclear Synthesis, No. 2, 57-64 (1980).
2. B. A. Kolovandin, "Stability of flow of a conducting liquid with free surface in the presence of electric and magnetic fields," Prikl. Mekh., 1, 95-105 (1965).
3. B. A. Kolovandin, "Stability of flow of electrically conducting liquid with free surface," in: Industrial Electromagnetic Hydrodynamics, Moscow (1967), pp. 368-376.
4. V. A. Bernshtam, S. V. Kozyrev, and A. I. Él'kin, "Stability of flow of conducting liqui film in longitudinal magnetic field," Magn. Gidrodin., No. 2, 37-41 (1981).
5. Hsien Din-Yu, "Stability of a conducting fluid flowing down an inclined plane in a magnetic field," Phys. Fluids, 8, No. 10, 1785-1791 (1965).
6. S. V. Kozyrev and A. I. Él'kin, "Stability of flow of thin films of electrically conducting liquid in crossed magnetic and electric fields," Magn. Gidrodin., No. 4, 57-60 (1981).
7. Yu. P. Ladikov, "Stability of flow of conducting liquid flowing along an inclined plane in the presence of a magnetic field," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 1, 3-8 (1966).
8. C. S. Yih, "Stability of liquid flow down an inclined plane," Phys. Fluids, 6, No. 3, 321-334 (1963).
9. I. M. Crowley, "Effect of electromagnetic force on the stability of liquid films," Ind. Eng. Chem., 6, No. 2, 243-246 (1967).