STUDY OF THE STABILITY OF A SYSTEM OF LIQUID-METAL JETS WITH CURRENTS

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Introduction. The study [1] examined a special liquid-metal system formed by a series of liquid-metal jets spaced uniformly in a circle with currents flowing in opposite directions in adjacent jets (Fig. 1). The interaction of the currents with the intrinsic field creates forces which result in motion of the jets in the radial direction and their deformation. Different types of MHD-instabilities may also occur.

These factors determine the limiting lifetime of the jet system as a functioning liquid structure and, thus, the maximum duration of the pulse received from the electrical power source.

1. Magnetic Field and Evaluation of the Time of Dispersion of the Jets in the Radial Direction. We will assume that the jets are infinitely long and that their cross section is invariant over their height. In this case, the problem for the vector potential of the magnetic field is formulated as follows

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 A}{\partial \varphi^2}=-\mu_0 j, \tag{1}$$

$$j=\begin{cases}(-1)^{m+1}j_0, & (r,\varphi)\in G_m,\\ 0, & (r,\varphi)\,\overline{\in}\, G_m, \quad m=1,2,\ldots,2p.\end{cases}$$

Here j_0 is the current density in the jet; G_m is the cross section of the m-th jet; p is the number of pairs of jets.

The magnetic field is found from the formulas

$$B_r=\frac{1}{r}\frac{\partial A}{\partial \varphi},\quad B_\varphi=-\frac{\partial A}{\partial r}. \tag{2}$$

If the boundaries of the jets coincide with the coordinate axes in a cylindrical coordinate system (Fig. 1), then we write the boundary conditions in the form

$$\left.\frac{\partial A}{\partial \varphi}\right|_{\varphi=0}=A|_{\varphi=\alpha}=A|_{r=0}=0,\quad A|_{r\to\infty}\to 0. \tag{3}$$

The solution of Eq. (1) should satisfy boundary conditions (3) and conditions of continuity of the vector potential and its derivative.

With allowance for (2), the solution of (1) leads to the following for the induction of the magnetic field:

$$B_r=\sum_n B_n{}^r \sin k_n\varphi,\quad B_\varphi=\sum_n B_n{}^\varphi \cos k_n\varphi,$$
$$B_n{}^r=B_n{}^\varphi=\frac{\mu_0 J_n}{2(k_n-2)}\left[r_2(r/r_2)^{k_n-1}-r_1(r/r_1)^{k_n-1}\right],\quad 0<r<r_1, \tag{4}$$

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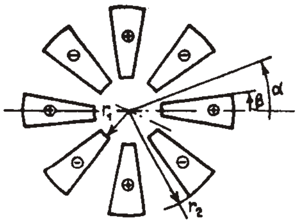

Fig. 1. Design sketch.

$$B_n^r=\mu_0 J_n\left[\frac{r_2}{2(k_n-2)}(r/r_2)^{k_n-1}+\frac{r_1}{2(k_n+2)}(r_1/r)^{k_n+1}-\frac{k_n r}{k_n^2-4}\right],$$

$$B_n^\varphi=\mu_0 J_n\left[\frac{r_2}{2(k_n-2)}(r/r_2)^{k_n-1}-\frac{r_1}{2(k_n+2)}(r_1/r)^{k_n+1}-\frac{2r}{k_n^2-4}\right],$$

$$r_1<r<r_2,$$

$$B_n^r=-B_n^\varphi=\frac{\mu_0 J_n}{2(k_n+2)}\left[r_1(r_1/r)^{k_n+1}-r_2(r_2/r)^{k_n+1}\right],\quad r>r_2.$$

Here, $J_n = (2j_0/\alpha k_n)\sin k_n\beta$, $\alpha = \pi/2p$; $\gamma = \beta/\alpha$ is the space factor of the blanket-magnet, $k_n = 2p(n + {}^1/_2)$, $n = 0, 1, 2, \ldots$.

In the case of jets of arbitrary cross section, the solution has the form [2]

$$A(M)=\frac{\mu_0}{2\pi}\int_G j(N)\ln\frac{1}{r_{MN}}\,dS_N. \tag{5}$$

Here, G is the region occupied by the jets in the plane xy; M is a point in this plane.

The vector potential of the system of current-carrying jets can be found by superposing the vector potentials of the individual jets. Assuming that the current density is constant and equal to j_0, we write the following for the vector potential of one jet:

$$A(M)=\frac{\mu_0 j_0}{2\pi}\int_G \ln\frac{1}{r_{MN}}\,dS_N. \tag{6}$$

We use Green's formula to change over in Eq. (6) to a contour integral enveloping the cross section G of the jet:

$$A(M)=\frac{\mu_0 j_0}{4\pi}\oint\left[(x_M-x_N)(\ln r_{MN}-0{,}5)\,dy_N-(y_M-y_N)(\ln r_{MN}-0{,}5)\,dx_N\right].$$

Having represented the contour bounding the conductor in the form of a closed broken line with the coordinates $N_i = (x_i, y_i)$ and having calculated the integrals over segments of the straight lines, we obtain the following for the vector potential and the components of the induction of the magnetic field:

$$A(M)=\frac{\mu_0 j_0}{8\pi}\sum_i (K_i-1)D_i,\quad B_x(M)=-\frac{\mu_0 j_0}{4\pi}\sum_i K_i\Delta x_i,$$

$$B_y(M)=-\frac{\mu_0 j_0}{4\pi}\sum_i K_i\Delta y_i.$$

Here

$$\Delta x_i=x_{i+1}-x_i,\quad \Delta y_i=y_{i+1}-y_i,\quad \mathbf{r}_i=(\Delta x_i,\Delta y_i),$$

$$D_i=(x_M-x_i)\Delta y_i-(y_M-y_i)\Delta x_i,\quad \mathbf{R}_i=(x_i-x_M,\ y_i-y_M),$$

$$K_i=r_i^{-2}\left[(\mathbf{R}_{i+1},\mathbf{r}_i)\ln R_{i+1}^2-(\mathbf{R}_i,\mathbf{r}_i)\ln R_i^2+2D_i\left(\operatorname{arctg}\frac{(\mathbf{R}_{i+1},\mathbf{r}_i)}{D_i}-\operatorname{arctg}\frac{(\mathbf{R}_i,\mathbf{r}_i)}{D_i}\right)-2\right].$$

Figure 2a shows the dependence of the induction of the magnetic field on the radius along a line of force going to infinity. Induction was calculated from Eq. (4) for the case of radially positioned jets, and a comparison is made between the cases of curved

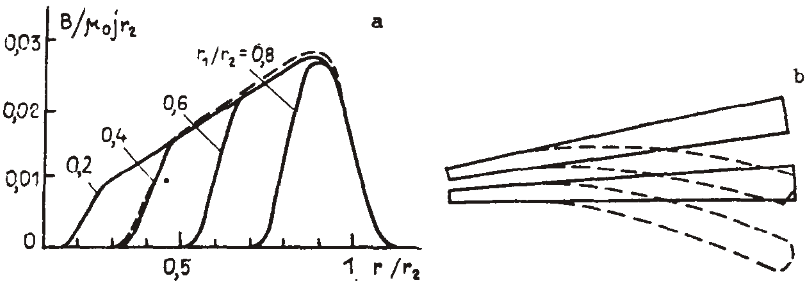

Fig. 2. Dependence of the induction of the magnetic field on the radius with different r_1/r_2 and jet geometries.

(dashed lines) and radial jets. The corresponding geometries are shown in Fig. 2b. The character of the distribution of the field is the same in both cases. The distribution of induction does not have a minimum over the radius.

We find the force acting over a unit height of an individual jet in the form

$$F_r = \int_G [\mathbf{j}\times\mathbf{B}]\,dS.$$

Having used Eq. (4), we obtain:

$$F_r = -\mu_0 j_0^2 \sum_n \frac{2}{\alpha k_n}\sin k_n\beta \left\{ \frac{\sin(k_n-1)\beta}{(k_n-1)(k_n-2)} \left[\frac{(1-(r_1/r_2)^{k_n+1}}{k_n+1} - \frac{1-(r_1/r_2)^3}{3} \right] + \right.$$

$$\left. + \frac{\sin(k_n+1)\beta}{(k_n+1)(k_n+2)} \left[\frac{1-(r_1/r_2)^3}{3} - (r_1/r_2)^3 \frac{1-(r_1/r_2)^{k_n-1}}{k_n-1} \right] \right\}.$$

The time of displacement of the jet as a whole over the distance δ in the radial direction is evaluated from the formula

$$t_1 \simeq \sqrt{\frac{2\delta\rho S}{F_r}}.$$

For jets of lead with the parameters

$$r_1 = 0{,}5\,\text{m};\quad r_2 = 1{,}5\,\text{m};\quad \gamma = 0{,}5;\quad p = 24;\quad B_r = 1\ \text{T};\quad \delta = 1\ \text{cm}$$

we obtain the characteristic time $t_1 \simeq 20$ msec.

2. Solution for Jets Infinite in the x and y Directions. We will examine stability against perturbations of the following type for the theoretical model shown in Fig. 3

$$\dot{\xi} = \xi(z)\exp i(\mathbf{kr} - \omega t), \tag{7}$$

where $\mathbf{k} = \mathbf{i}k_x + \mathbf{j}k_y$ is the wave vector; ω is the frequency.

The case $k_y = 0$ corresponds to kink instability, while the case $k_x = 0$ corresponds to decay of each of the current-carrying jets into individual smaller jets.

For the sake of generality, we will also consider the possibility of instability due to accelerations of the boundary. Henceforth, we will use the approach in [3], meanwhile allowing that the condition $\mu_0\sigma\Omega a^2 \ll 1$, is observed. Here, Ω is the equivalent frequency of the current pulse. Thus, we can ignore the skin effect and assume that the undisturbed currents are uniform.

In the general case, perturbations propagating over different surfaces of the jets are phase-shifted relative to each other by different angles. We will restrict ourselves to the limiting cases in which the phase shift of the perturbations on one jet is 0 or 180° and the difference of the phases of perturbations θ_2 on surfaces facing each other has the same values. In all, we obtain four variants of perturbations. We use (7) to express perturbations of all quantities, assuming the perturbations to be small compared to the undisturbed quantities.

In the approximation used in [4], the equations for the perturbation of the induction of the magnetic field are written in the form

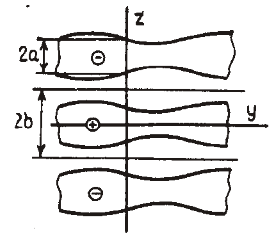

Fig. 3

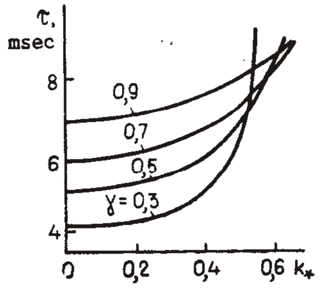

Fig. 4

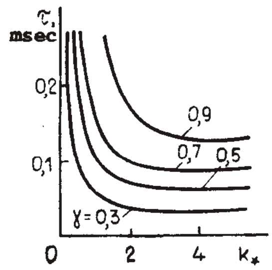

Fig. 5

Fig. 3. Theoretical model of instability of the current-carrying jet.

Fig. 4. Dependence of the time of disintegration of a jet by a kink instability on the wave number.

Fig. 5. Dependence of the time of disintegration of a jet by its subdivision into small jets on the wave number.

$$\frac{d^2\dot{\mathbf{B}}_1}{dz^2}-k^2\dot{\mathbf{B}}_1=0;\quad \mathbf{B}_1{}^e=\nabla\Psi_1,\quad \frac{d^2\dot{\Psi}_1}{dz^2}-k^2\dot{\Psi}_1=0. \tag{8),(9}$$

Equation (8) is valid in the liquid metal, while Eq. (9) is valid outside the metal.

The boundary conditions for Eqs. (8) and (9) follow from the conditions of continuity of the magnetic field in the transition across the boundary of the liquid metal

$$\dot{B}_{1x}(a)=\dot{B}_{1x}{}^e(a),\quad B_{1y}(a)+\dot{\xi}_z(a)B_0/a=B_{1y}{}^e(a),\quad \dot{B}_{1z}(a)=\dot{B}_{1z}{}^e(a)$$

and the conditions of symmetry for the corresponding cases:

$$\dot{B}_{1x}(0)=\dot{B}_{1y}(0)=\dot{B}_{1z}{}'(0)=0,\quad (1,2),$$
$$\dot{B}_{1x}{}'(0)=\dot{B}_{1y}{}'(0)=\dot{B}_{1z}(0)=0,\quad (3,4),$$
$$\dot{B}_{1x}{}^e(b)=\dot{B}_{1y}{}^e(b)=\dot{B}_{1z}{}^{e\prime}(b)=0,\quad (2,4),$$
$$\dot{B}_{1x}{}^{e\prime}(b)=\dot{B}_{1y}{}^{e\prime}(b)=\dot{B}_{1z}{}^e(b)=0,\quad (1,3),$$

where $B_0 = \mu_0 j_0 a$. Here, 1 corresponds to $\theta_1 = 180°$, $\theta_2 = 180°$; 2, $\theta_1 = 180°$, $\theta_2 = 0$; 3, $\theta_1 = 0$, $\theta_2 = 180°$; 4, $\theta_1 = 0$, $\theta_2 = 0$.

The boundary conditions for Ψ_1 have the form

$$\dot{\Psi}_1(b)=0,\quad (2,4),\quad \dot{\Psi}_1{}'(b)=0,\quad (1,3).$$

The equations of motion and continuity for the perturbations are written in the form

$$-\rho\omega^2\dot{\xi}_x=-ik_x\dot{p}_1+\dot{F}_{1x},\quad -\rho\omega^2\dot{\xi}_y=-ik_y\dot{p}_1+\dot{F}_{1y},$$
$$-\rho\omega^2\dot{\xi}_z=-\dot{p}_1{}'+\dot{F}_{1z},\quad ik_x\dot{\xi}_x+ik_y\dot{\xi}_y+\dot{\xi}_z{}'=0. \tag{10}$$

The expressions for the components of the perturbations of the electromagnetic force have the form

$$\dot{F}_{1x}=\frac{B_0 z}{\mu_0 a}(ik_y\dot{B}_{1x}-ik_x\dot{B}_{1y}),\quad \dot{F}_{1y}=\frac{B_0\dot{B}_{1z}}{\mu_0 a},$$

$$\dot{F}_{1z}=\frac{B_0}{\mu_0 a}[z(ik_y\dot{B}_{1z}-\dot{B}_{1y}{}')-\dot{B}_{1y}].$$

It follows from system (10) that:

$$\dot{\xi}_z{}''-k^2\dot{\xi}_z=0,\quad \rho\omega^2=\frac{1}{\dot{\xi}_z{}'(a)}(k^2\dot{p}_1(a)+i\mathbf{k}\cdot\dot{\mathbf{F}}_1). \tag{11),(12}$$

The boundary conditions of Eq. (11) have the form

$$\dot{\xi}_z(a)=\dot{\xi}_z(-a),\quad \dot{\xi}_z(a)=-\dot{\xi}_z(-a). \tag{13),(14}$$

Condition (13) is valid in the absence of a phase shift on the surface of one jet, while condition (14) is valid with a phase shift of 180°.

The solutions for the corresponding cases are written in the form

$$\dot{\xi}_z(z)=\dot{\xi}_z(a)\frac{\operatorname{ch} kz}{\operatorname{ch} ka},\quad \dot{\xi}_z(z)=\dot{\xi}_z(a)\frac{\operatorname{sh} kz}{\operatorname{sh} ka}.$$

The expression for the perturbation of pressure on the boundary is

$$\dot{p}_1(a)=[\sigma_s k^2-G(a)-F_0(a)]\dot{\xi}_z(a).$$

Here, σ_s is the surface tension,

$$G(a)=-\rho\frac{dv_{0z}}{dt},\quad F_0(a)=-\frac{B_0^2}{\mu_0 a}.$$

Inserting the expressions found for $p_1(a)$, $\xi_z'(a)$ and $\dot{F}_1(a)$ into Eq. (12) and changing over to dimensionless variables, we obtain dispersion relations for different types of perturbations

$$\omega_*^2=\operatorname{th} k_*\left[\left(\frac{k_y}{k}\right)^2\left(k_*-\frac{\operatorname{sh} k_*(b_*-1)\operatorname{ch} k_*}{\operatorname{ch} k_* b_*}\right)+\omega_{*1}^2\right],$$

$$\omega_*^2=\operatorname{th} k_*\left[\left(\frac{k_y}{k}\right)^2\left(k_*-\frac{\operatorname{ch} k_*(b_*-1)\operatorname{ch} k_*}{\operatorname{sh} k_* b_*}\right)+\omega_{*1}^2\right],$$

$$\omega_*^2=\operatorname{cth} k_*\left[\left(\frac{k_y}{k}\right)^2\left(k_*-\frac{\operatorname{ch} k_*(b_*-1)\operatorname{sh} k_*}{\operatorname{ch} k_* b_*}\right)+\omega_{*1}^2\right],$$

$$\omega_*^2=\operatorname{cth} k_*\left[\left(\frac{k_y}{k}\right)^2\left(k_*-\frac{\operatorname{sh} k_*(b_*-1)\operatorname{sh} k_*}{\operatorname{sh} k_* b_*}\right)+\omega_{*1}^2\right].$$

Here, $\omega_* = \omega/\omega_0$, $\omega_0^2 = B_0^2/\mu_0\rho a^2$, $k_* = ka$, $b_* = b/a$, $\omega_{*1}^2 = k_*(Ak_*^2 - B)$, $A = \sigma_s/a^3\omega_0^2$, $B = G(a)/a\omega_0^2$.

3. Solution for Jets Bounded in the y Direction. In the above case of jets oriented in the x direction, kinks did not lead to strengthening of the magnetic field and, thus, the onset of instability. The study [4] examined a case of practical interest whereby the cross sections of the jets are highly elongated but bounded in the y direction, and small perturbations of the following form propagate along the broad sides of the jets in the x direction

$$\xi=\dot{\xi}(z)\exp(\omega t+ikx). \tag{15}$$

In this case, due to shortening of the lines of force with the occurrence of perturbations of type (15), kink instabilities take place with an increment:

$$\omega_*^2=\frac{k_*}{b_*\Delta_{*1}\operatorname{sh} k_*}\sum_{n=0}^{\infty}\left[\mu_n^2\frac{\operatorname{sh}^2 k_*}{\operatorname{ch} k_*}\left(\frac{M_n\sin^2\mu_n}{\mu_n^3 b_n^2}-\frac{N_n\cos^2\mu_n}{a_n^5}\right)+\right.$$

$$\left.+2k_*\mu_n\operatorname{sh} k_*\sin\mu_n\cos\mu_n\left(\frac{M_n}{\mu_n^3 b_n^2}+\frac{N_n}{a_n^5}\right)-k_*^2\frac{\sin^2\mu_n}{\operatorname{ch} k_*}\left(\frac{2M_n}{\mu_n^3 b_n^2}+\frac{N_n\operatorname{ch}^2 k}{a_n^5}\right)\right].$$

Here, $\omega_0^2 = \mu_0 j_0^2/\rho$, $\mu_n = \pi b_*^{-1}(n + 1/2)$, $a_n^2 = \mu_n^2 + k_*^2$, $b_n^2 = \mu_n^2 + 4k_*^2$, $M_n = 1 - \exp(-2\mu_n\Delta_{*1})$, $N_n = 1 - \exp(-2a_n\Delta_{*1})$; $2\Delta_1$ is the size of the jet in the y direction.

4. Results of Numerical Calculations. Figures 4 and 5 show the time of disintegration of a lead system of jets with $r_1 = 0.5$ m, $r_2 = 1.5$ m. This system contains 48 jets and is characterized by different space factors γ and a fixed field $B_0 = 1$ T. Its disintegration is caused by instabilities of both of the types discussed in the previous sections.

Instabilities of type 2 (Fig. 4) are the most dangerous. Meanwhile, the characteristic lifetime $\tau = \omega^{-1}$ of the jet system depends weakly on the space factor $\gamma = a/b$ of the chamber enclosing the liquid-metal jets and has a value close to $\tau = 5$ msec. Instability is greater in this system than for an isolated jet. This can be attributed to intensification of the perturbations of the magnetic field in case 2, leading to an increase in the magnitude of the periodic and y-alternating component of the electromagnetic force in the y direction and consequent transverse swellings of the electrically conducting liquid. For kink instabilities described in the previous section, τ = 50 msec.

Conclusions. Analytic expressions were obtained to describe the magnetic field of a system composed of a series of liquid-metal jets distributed uniformly in a circle and in which currents flow in adjacent jets in opposite directions.

2. The magnetic fields for radial and slightly curved jets were calculated. It was shown that the character of distribution of the field over the radius is the same in both cases. Meanwhile, the field distribution does not have a minimum over the radius.

3. Analytic relations were obtained that make it possible to evaluate the time of radial dispersion of the jets and to calculate the increments of different MHD-instabilities (MHD-instabilities leading to division into smaller individual jets, kink instabilities).

4. The lifetime of the jet system evaluated from the relations is $\tau \simeq$ 5-10 msec for the given parameters.

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