HYDRODYNAMIC INSTABILITY OF A UNIFORM VELOCITY DISTRIBUTION IN A CYLINDRICAL PUMP

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UDC 621.313.53:537.84

Using the method of small perturbations, stability is investigated for a uniform flow of a viscous, incompressible electrically conducting liquid in a straight infinitely long coaxial channel having a radius R exposed to the action of a magnetic field moving along the x axis. Within the framework of a one-dimensional "narrow strip" model, this problem has been previously considered in [1, 2]. In the present work, the possibility of a two-dimensional motion of the liquid is considered.

The same assumptions are made relative to the geometrical channel dimensions and the character of the unperturbed flow as in [1, 2], i.e., $\delta/R \ll 1$, $\delta/\tau \ll 1$ (δ being the channel height and $\tau = \pi/\alpha$ the polar pitch). Assuming the validity of the small gap approximation, we can write the induction equation for the sole field component in dimensionless form:

$$\partial b/\partial t + u\partial b/\partial x + v\partial b/\partial y = \mathrm{Rm}^{-1}\Delta b - u\partial B_0/\partial x - \partial B_0/\partial t, \tag{1}$$

where x is the coordinate in the direction of the uniform liquid motion, $y = R\varphi$, φ is the polar angle, R is the mean channel radius, and u and v are the velocity components along the x and y axes, respectively.

Equation (1) coincides with Eq. (1) in [2] with the exception of the additional term in the convective term. The following time, length and velocity values have been used in the nondimensionalizing: $T = 1/\omega$, $L = 1/\alpha$, and $U = \omega/\alpha$; then $\mathrm{Rm} = \sigma\mu_0\omega/\alpha^2$. The amplitude of the moving wave of the external field B_0 is taken as the characteristic value of the magnetic field induction. The phase relationships are selected in such manner, as to make the dimensionless function $B_0(x, t)$ in (1) to equal $B_0(x, t) = \cos\theta$, with $\theta = t - x$. Equation (1) then assumes the form:

$$\partial b/\partial t + u\partial b/\partial x + v\partial b/\partial y = \mathrm{Rm}^{-1}\Delta b + (1-u)\sin\theta. \tag{2}$$

We write the equations of motion in the form

$$\partial u/\partial t + u\partial u/\partial x + v\partial u/\partial y = -\partial p/\partial x + \mathrm{Re}^{-1}\Delta u - C_x u|V| - \mathrm{Al}((B_0+b)\partial b/\partial x), \tag{3}$$

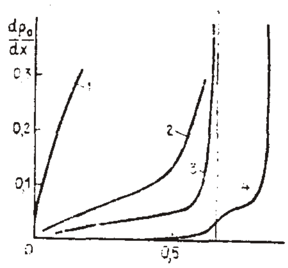
$$\partial v/\partial t + u\partial v/\partial x + v\partial v/\partial y = -\partial p/\partial y + \mathrm{Re}^{-1}\Delta v - C_y v|V| - \mathrm{Al}((B_0+b)\partial b/\partial y), \tag{4}$$

where p is the pressure, Re the Reynolds number, Al is the Alfven number, $|V| = \sqrt{u^2+v^2}$, and C_x and C_y are coefficients in the Chezy's frictional law. The velocity components are related by the continuity equation $\partial u/\partial x + \partial v/\partial y = 0$.

Equations (2) to (4) admit a solution $u = u_0 = \mathrm{const}$, $v = 0$, $b = b_0(x, t) = \mathrm{Rm}_s(1 + \mathrm{Rm}_s^2)^{-1}(\sin\theta - \mathrm{Rm}_s\cos\theta)$, and $\mathrm{Rm}_s = \mathrm{Rm}(1 - u_0)$. Substituting these equations into (3), the following expression is obtained for the pressure gradient:

$$\partial p_0/\partial x = 0.5\,\mathrm{Al}\,\mathrm{Rm}_s(1+\mathrm{Rm}_s^2)^{-1} - C|u_0|u_0 + \mathrm{Rm}_s(1+\mathrm{Rm}_s^2)^{-2}(\mathrm{Rm}_s\sin 2\theta + (1-\mathrm{Rm}_s^2)\cos 2\theta/2). \tag{5}$$

Translated from Magnitnaya Gidrodinamika, No. 1, pp. 121-124, January-March, 1987. Original article submitted May 28, 1986; revision submitted September 22, 1986.



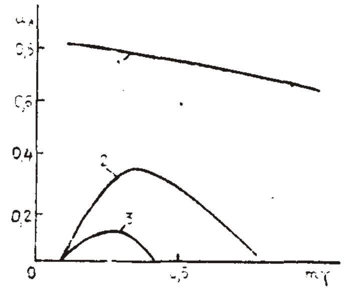


Fig. 1 Fig. 2

From here on we shall characterize (analogously to [1]) the pressure gradient by the constant portion of Eq. (5).

We next consider the perturbed motion in the form $u = u_0 + u'$, $v = v'$. After substituting these expressions into Eqs. (2)-(4) and applying the standard linearization method, we obtain linear differential equations for perturbations u', v', b', and p'. We introduce vorticity ω' and the stream function ψ' according to formulas $\omega' = \partial u'/\partial y - \partial v'/\partial x$, $u' = \partial\psi'/\partial y$ and $v' = -\partial\psi'/\partial x$, Eliminating the pressure perturbation p', we obtain

$$\partial b'/\partial t + u_0 \partial b'/\partial x = \mathrm{Rm}^{-1}\Delta b' - (1+\mathrm{Rm}_s^2)^{-1}(\sin\theta - \mathrm{Rm}_s\cos\theta)\partial\psi'/\partial y, \tag{6}$$

$$\partial\omega'/\partial t + u_0\partial\omega'/\partial x = \mathrm{Re}^{-1}\Delta\omega' - 2C|u_0|\omega' + \mathrm{Al}\sin\theta\partial b'/\partial y,\quad \Delta\psi' = \omega'. \qquad (7),\ (8)$$

We consider perturbations that are periodical in the y-coordinate:

$$\omega' = \tilde{\omega}(x, t)e^{im\gamma y},\quad b' = \tilde{b}(x, t)e^{im\gamma y},\quad \psi' = \tilde{\psi}(x, t)e^{im\gamma y}, \tag{9}$$

where $m = 1, 2, \ldots$, $\gamma = \tau/a$, $a = \pi R$ is the channel semiwidth. By substituting these expressions into (6)-(8), and at the same time changing from variables as functions of (x, t) to those of (θ, t), we obtain:

$$\partial\tilde{\omega}/\partial t = -s\partial\tilde{\omega}/\partial\theta + \mathrm{Re}^{-1}(\partial^2\tilde{\omega}/\partial\theta^2 - (m\gamma)^2\tilde{\omega}) - 2C|u_0|\tilde{\omega} + im\gamma\,\mathrm{Al}\sin\theta\tilde{b},\quad s = 1 - u_0, \tag{10}$$

$$\partial\tilde{b}/\partial t = -s\partial\tilde{b}/\partial\theta + \mathrm{Rm}^{-1}(\partial^2\tilde{b}/\partial\theta^2 - (m\gamma)^2\tilde{b}) - im\gamma(1+\mathrm{Rm}_s^2)^{-1}(\sin\theta - \mathrm{Rm}_s\cos\theta)\tilde{\psi}, \tag{11}$$

$$\partial^2\tilde{\psi}/\partial\theta^2 - (m\gamma)^2\tilde{\psi} = \tilde{\omega}. \tag{12}$$

We consider approximate solutions in the form

$$\tilde{\omega}(\theta, t) = \omega_0(t) + \sum_{k=1}^{N}(\omega_{k1}(t)\cos k\theta + \omega_{k2}(t)\sin k\theta), \tag{13}$$

$$\tilde{b}(\theta, t) = b_0(t) + \sum_{k=1}^{N}(b_{k1}(t)\cos k\theta + b_{k2}(t)\sin k\theta), \tag{14}$$

$$\tilde{\psi}(\theta, t) = \psi_0(t) + \sum_{k=1}^{N}(\psi_{k1}(t)\cos k\theta + \varphi_{k2}(t)\sin k\theta). \tag{15}$$

If in these equations we limit ourselves to the first terms, we obtain perturbations within the framework of the narrow strip model. By substitution of Eqs. (13)-(15) into (10)-(12) and by comparison of corresponding terms in sin kθ and cos kθ, we obtain a system of ordinary linear differential equations with constant corfficients. The question of the stability of motion is is thus transposed into a question concerning the existence of eigenvalues containing a real part in the coefficient matrix. For each specific selection of the Re, Rm, Al, mγ, u_0, and C the eigenvalues were found using the ATEIG subroutine included in the software of the EC computer. The complex matrix approximating Eqs. (10)-(12) was found to be a real matrix of doubled dimension. To solve for the motion arising after the loss of stability, an eigenvector was evaluated which corresponds to the eigenvalue λ_0 having the maximum real part. For purely real λ_0 the reverse iteration algorithm [3, p. 166] was used, while for complex λ_0 its

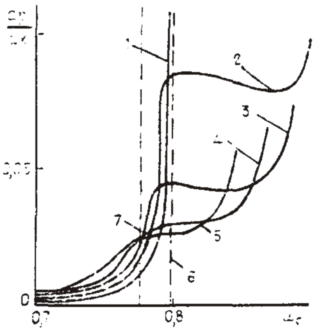


Fig. 3

modification [4, p. 275] was employed. The computations were carried out mainly for N = 3 and N = 5. Increasing N to 7 and 10 did not effect the results, some of which are presented in Figs. 1-3.

In Figs. 1 and 3 velocity u_0 is plotted along the abscissa and the pressure gradient $\partial p/\partial x$ along the ordinate. Only the constant term of the right hand portion in (5) has been taken into account. Shown in these figures are neutral curves $\mathrm{Real}(\lambda_0) = 0$ dividing the regions of stable and unstable uniform flow.

We next compare the results of the present work with the well-known model of Gailitis and Lielausis [1]. We limit ourselves to the consideration of the pumping regime $0 \leq u_0 \leq 1$. In this case the conclusions reached in [1] come down basically to the following: 1) an instability regime is possible only at Rm satisfying the condition $\mathrm{Rm} \geqslant \sqrt{1+(m\gamma)^2}$; 2) in the $(u_0, \partial p/\partial x)$ plane the neutral curve originates at the origin of coordinates and has a vertical asymptote whose position is determined by the equation (see (33) in [1]) $s\mathrm{Rm} = \sqrt{1+(m\gamma)^2}$, $s = 1 - u_0$; 3) the most destabilizing perturbations are those with m = 1.

Figure 1 shows neutral curves in the $(u_0, \partial p/\partial/x)$ plane for Rm = 1 (1 and 2) and Rm = 3 (3 and 4). In this figure, plotted as a dash-dot line, is the straight line $u_0 = 1 - \sqrt{1+(m\gamma)^2}/\mathrm{Rm}$ for Rm = 3 and $m\gamma = 0.3$. Curve 3 is drawn for parameters Rm = 3, $m\gamma = 0.3$ Re = 1000 and C = 0.04 based upon the results of the present work. It is evident that good agreement exists with the results of [1]. The eigenvalues corresponding to the most destabilizing perturbations in the vicinity of the neutral curve are purely real. It follows therefore that the transition proceeds to a new stationary state, as proposed in the theory of [1]. Curve 4 corresponds to the same values of parameters Rm and $m\gamma$; however, here Re = 10,000 and C = 0.004. The instability region is substantially larger, part of it situated to the right of the dash-dot straight line $u_0 = 1 - \sqrt{1+(m\gamma)^2}/\mathrm{Rm}$. To the right of it, eigenvalues, whose real part revert to zero on the neutral curve, have a nonzero imaginary part. Consequently, after the loss of stability in a uniform flow an oscillating motion is produced. Curves 1 and 2 are drawn for Rm = 1, Re = 10,000, C = 0.004 and $m\gamma = 0.1$ (1) and $m\gamma = 0.3$ (2). In this case, a larger $m\gamma$ value corresponds to a larger instability region. It follows then that stability can be lost with m > 1.

Figure 2 shows the dependence of the average velocity u_0 at which stability is lost on $m\gamma$ for Al = 4, Re = 1000 and C = 0.04. Curves 1-3 correspond to Rm = 5, 1, and 0.5. For Rm = 1 and Rm = 0.5 this dependence displays a nonmonotonous character. In Fig. 3 the boundries of the instability region are drawn for Rm = 5, Re = 10,000 and C = 0.004. Curves 1-5 correspond to the parameter $m\gamma$ having the values 0.1, 0.2, 0.3, 0.4, and 0.5. Dashed lines show vertical asymptotes from theory [1] for Rm = 5, $m\gamma = 0.1$ (6) and $m\gamma = 0.5$ (7). The curve 1 ($m\gamma = 0.1$) approaches its vertical asymptote 6 beyond the boundaries of the graph; at about $\partial p/\partial x = 0.35$ it turns to the right and crosses the asymptote. The vertical asymptotes drawn in accordance with [1] divide each neutral curve into two parts. In one of these, corresponding to a large slip, a transition takes place to a new stationary regime while in the other, situated in the $(u_0, \partial p/\partial x)$ plane to the right of the asymptote, loss of stability in uniform flows produces an oscillating motion.

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