THE EFFECT OF A MAGNETIC FIELD ON THE THERMOCAPILLARY MOTION IN A CONDUCTING DROPLET

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The possibility [1] of replacing fixed divertor screens in thermonuclear synthesis equipment by a sheet of free-falling liquid-metal droplets has recently come under discussion. This sheet of droplets is subjected to powerful one-sided heating, setting up a surface-tension gradient. We are thus confronted with the problem of estimating the possible thermocapillary motions.

The research at hand with respect to thermocapillary motion was conducted on droplets not capable of conducting electricity [2] and the results of this work can be used in an examination of metal droplets, but only in the absence of a magnetic field, or more precisely, at low values for the Hartmann number (Ha << 1). The heating anticipated within this sheet ($\sim$1 MW/m^2) without a field must lead to substantial intradrop velocities u $\sim$ 1 m/sec and, as a consequence, to convective heat transfer within the droplet. We might even be confronted with fragmentation of the droplet.

In actual fact, the sheet functions within a powerful magnetic field, and the reverse condition is valid, namely, Ha >> 1. The field decelerates the motion, and the cited phenomena weaken. To clarify the quantitative effects, we have below solved the hydrodynamic problem of thermocapillary motion for the case in which Ha >> 1.

Formulation of the Problem. Let there be a spherical droplet (see Fig. 1) in a void, with a nonuniform distribution of temperatures $T(\vartheta, \varphi)$ maintained at the surface of the droplet. Consequently, the surface tension will also be nonuniform:

$$\alpha(\vartheta, \varphi) \approx \alpha(T_0) + (T(\vartheta, \varphi) - T_0)\, d\alpha/dT;$$

here T_0 is the mean temperature; r, ϑ, and φ are the spherical coordinates. When $|\alpha(\vartheta, \varphi) - \alpha(T_0)| << \alpha(T_0)$, the surface of the droplet remains approximately spherical. Thermocapillary motion is set up within the droplet, and this motion follows the boundary condition

$$\mathrm{grad}_{\vartheta,\varphi}\alpha(\vartheta, \varphi) = \eta r \frac{\partial}{\partial r}(u_{\vartheta,\varphi}/r)\,|_{r=R}. \quad (1)$$

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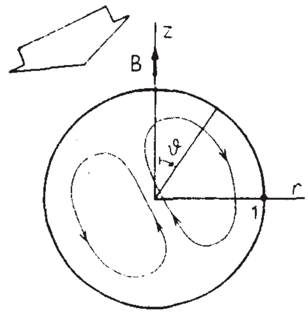

Fig. 1. Formulation of the problem. Heating proceeds in the direction of the broad arrow; the movements within the droplets are illustrated by the closed curves.

When an external magnetic field is impressed, boundary condition (1) shows no change. At the Stokes limit Re << max $(\mathrm{Ha}^2, 1)$ the field B, the viscosity η, and the electrical conductivity σ enter the equations of motion only through the Hartmann number $\mathrm{Ha} = BR(\sigma/\eta)^{1/2}$. The magnetic Reynolds number is neglected. The drop radius (R = 1) is taken as the unit of length. In addition to the above-cited spherical coordinate system, we also use a cylindrical coordinate system ρ, φ, z whose z axis lies in the direction of the external uniform magnetic field **B**. The vector components perpendicular to **B** will be marked with the subscript $\perp$ in the following. In addition, we have the notation $s = \sin\vartheta$, $c = \cos\vartheta$, $\Delta_\perp = \Delta - \partial^2/\partial z^2$.

Eliminating pressure and the electric potential from the MHD equation is accomplished by the Gotoh substitution

$$\mathbf{u}_\perp = \mathrm{grad}_\perp P^+ + \mathrm{rot}\,(\Phi^+\mathbf{e}^z); \quad u_z = \partial P^+/\partial z - \mathrm{Ha}\,P^-, \tag{2}$$

which represents the determination of six quantities, i.e., three velocity components **u** and three electric-current density components **j**, thus reducing to the solution of four equations

$$(\Delta - \mathrm{Ha}\,\partial/\partial z)\,(P^1, \Phi^1) = 0; \quad (\Delta + \mathrm{Ha}\,\partial/\partial z)\,(P^2, \Phi^2) = 0 \tag{3}$$

and to the calculation of four scalar functions $P^\pm = P^1 \pm P^2$; $\Phi^\pm = \Phi^1 \pm \Phi^2$.

According to [3] the current is equal to

$$\mathbf{j}_\perp = \mathrm{rot}\,(P^+\mathbf{e}^z) - \mathrm{Ha}^{-1}\,\mathrm{grad}_\perp\,\partial\Phi^-/\partial z; \quad j_z = \partial\Phi^+/\partial z - \mathrm{Ha}^{-1}\,\partial^2\Phi^-/\partial z^2.$$

Two boundary conditions (1) are inadequate for four functions. We need, additionally, the elimination of the velocity and electric-current components which intersect the surface:

$$u_r = 0\,|_{r=1}; \quad j_r = 0\,|_{r=1}. \tag{4}$$

If $\alpha(\vartheta, \varphi)$ is specified in (1) in the form of $\alpha(\vartheta, \varphi) = \Sigma\alpha^m(\vartheta)e^{im\varphi}$, then it is possible individually to calculate the motion engendered by each of the terms individually, and then simply to combine the results. In the following, the index m with $\alpha(\vartheta)$ will be omitted under the assumption that the sum contains only a single term, i.e., $\alpha(\vartheta)$, and that all of the unknown quantities are proportional to $\exp(im\varphi)$. In cylindrical coordinates

$$\begin{aligned} rj_r &= \mathrm{Ha}^{-1}\,(z\Delta_\perp - \rho\partial^2/\partial\rho\partial z)\,\Phi^- + imP^+; \quad ru_r = (\rho\partial/\partial\rho + z\partial/\partial z)\,P^+ - \\ &- \mathrm{Ha}\,zP^- + im\Phi^+; \quad ru_\vartheta = (z\partial/\partial\rho - \rho\partial/\partial z)\,P^+ + \mathrm{Ha}\,\rho P^- + imz/\rho P^+. \end{aligned} \tag{5}$$

The utilized differential operators are equal to

$$\begin{aligned} &z\Delta_\perp - \rho\partial^2/\partial\rho\partial z = (\partial/\partial\vartheta + z/\rho)\,\partial/\partial\rho - zm^2/\rho^2; \\ &\rho\partial/\partial\rho + z\partial/\partial z = r\partial/\partial r; \quad z\partial/\partial\rho - \rho\partial/\partial z = \partial/\partial\vartheta. \end{aligned} \tag{6}$$

Asymptotic Solutions for Large Hartmann Numbers (Ha >> 1). For a clear representation of the properties of Eqs. (3) it is a simple matter, mentally, to equate the Hartmann number Ha to the Peclet number and to equate $\Phi^{1,2}$ or $P^{1,2}$ to temperature. Then, each of the equations in (3) will be identical to the equation of heat conduction in a uniformly moving medium. In the case of Φ^1 and P^1 the imagined medium within a spherical cavity, with $r \leq 1$, moves in the direction of the z axis, while in the case of Φ^2 and P^2 it moves in the direction $-z$. This analogy, it goes without saying, is purely formal and bears no relationship to the actual transfer of heat that takes place within the droplet being examined here. However, it is useful in the construction of asymptotic solutions for Ha >> 1 [4]:

a) within a cavity for which r ≤ 1 the solution for $\Phi^1(\rho, \varphi, z)$ is fully defined by the value of Φ^1 at the r = 1 surface; this applies equally to Φ^2, P^1, and P^2;

b) for large Ha >> 1 the "heat" is transferred primarily in conjunction with the medium. The role of "heat conduction" is insignificant throughout virtually the entire volume of the droplet, and along the "streamlines" the "temperature" is almost constant and equal to the "temperature" of the lower wall

$$\Phi^1(\rho, \varphi, z) = \Phi^1[\rho, \varphi, -(1-\rho^2)^{1/2}] + O(\mathrm{Ha}^{-1});$$

c) a "temperature" boundary layer of thickness 1/(Ha c) is formed near the upper wall (more precisely, a Hartmann layer), in which the "temperature" sharply attains the "temperature" of the upper wall:

$$\Phi^1(\rho, \varphi, z) \approx \Phi^1[\rho, \varphi, -(1-\rho^2)^{1/2}] + \delta\Phi^1;$$
$$\delta\Phi^1 = \{\Phi^1[\rho, \varphi, (1-\rho^2)^{1/2}] - \Phi^1[\rho, \varphi, -(1-\rho^2)^{1/2}]\} \exp[\mathrm{Ha}\, c(r-1)]. \tag{7}$$

The shape of the Hartmann layer, proportional to exp [Ha c(r − 1)], is obtained through substitution of (7) into Eq. (3) and by retaining the term that is fundamental with respect to Ha. In the case of $\delta\Phi^1$ this is sufficient whereas in the case of δP^1 it becomes necessary to find a solution in the following approximation: the refined solution for the equation

$$(\Delta - M\partial/\partial z)\delta P^1 = 0 \tag{8}$$

will be sought in the form

$$\delta P^1 = \exp[a(r, c) + im\varphi]; \tag{9}$$

substitution of (9) into (8) yields

$$a_{rr} + a_r^2 + 2a_r/r + (1-c^2)(a_{cc} + a_c^2)/r^2 - 2ca_c/r^2 - m^2/[(1-c^2)r^2] -$$
$$-\mathrm{Ha}[ca_r + (1-c^2)a_c/r] = 0$$

(the notation is $a_r = \partial a/\partial r$, etc.). The first two terms of the solution are equal to

$$a_r = \mathrm{Ha}\, c - 2 + a_c(1-c^2)/c + O(\mathrm{Ha}^{-1}). \tag{10}$$

The fundamental Ha c term leads to (7). Expression (10), on the whole, yields the asymptotic relationship

$$(r\partial/\partial r - \mathrm{Ha}\, z)\delta P^1|_{r=1} \approx -(s\partial/\partial s + s^2/c^2 + 2)\delta P^1|_{r=1}. \tag{11}$$

All of this pertains equally to Φ^2 and P^2, and it is only the Hartmann layer that is formed at the lower surface of the droplet.

Flow within a Drop Heated Symmetrically Relative to the Field. In the case of symmetrical heating (m = 0) the pattern of the flow is quite simple. Maximum velocities arise within the Hartmann layer, where in accordance with (1)

$$u_\vartheta = (d\alpha/d\vartheta)\exp[|c|\mathrm{Ha}(r-1)]/(\eta|c|\mathrm{Ha}).$$

The total rate of substance flow through the entire thickness of the Hartmann layer is smaller by a factor of c Ha (here +/− refers to the upper/lower surfaces of the droplet):

$$q^{\pm}(\rho) = \pm(d\alpha/d\vartheta)/(\eta c^2\mathrm{Ha}^2). \tag{12}$$

The flow described in (12) closes within the main volume of the drop. The flow is reversed out of the uniform flow along z toward the axis and in the direction of the field, where it is linear with respect to z:

$$2u_\rho = -[q^+(\rho) + q^-(\rho)]/(1-\rho^2)^{1/2};$$
$$2u_z = (\rho^{-1}\partial/\partial\rho)[z\rho(q^+ + q^-)/(1-\rho^2)^{1/2} + \rho(q^+ - q^-)].$$

In terms of order of magnitude the velocity within the Hartmann layer is smaller by a factor of Ha, and smaller by a factor of Ha^2 within the primary volume, than is the case without a magnetic field.

Droplet Heated Nonsymmetrically with Respect to the Field (m ≠ 0). To reduce the number of simultaneously determined functions the arbitrary $\alpha(\vartheta)$ should be presented as the sum of the even $[\alpha(-\vartheta) = \alpha(\vartheta)]$ and odd parts. In the following we will present a solution for the even case, in which $P^1(\rho, z) + P^2(\rho, -z)$, $\Phi^1(\rho, z) = \Phi^2(\rho, -z)$.

In the upper half-drop the Hartmann layer δP, $\delta\Phi$ forms only P^1 and Φ^1. With the required accuracy here we have

$$\Phi^+ = \Phi_0 + \delta\Phi;\quad \Phi^- = \frac{z}{\mathrm{Ha}}\Delta\Phi_0 + \delta\Phi;\quad P^+ = P_0 + \delta P;\quad P^- = \frac{z}{\mathrm{Ha}}\Delta P_0 + \delta P. \tag{13}$$

In (13) all of the quantities are proportional to exp (im φ). Moreover, Φ_0 and P_0 are functions of ρ, while δP and $\delta\Phi$ are products of the slow functions of c and the Hartmann boundary-layer factor exp [c Ha(r − 1)].

With substitution of (13) into boundary conditions (1) and (4) the calculation of the derivatives of Φ_0 and P_0 presents no difficulties. In processing $\delta\Phi$ and δP the more significant terms in (5) are shortened. In order to obtain the truly principal terms it is necessary to employ differential operators in the form of (6) and for δP also to make use of the asymptotic identity (11). The principal terms (r = 1) follow out of the boundary conditions:

$$\begin{aligned} &\delta P = (\eta s c)^{-1}\,\mathrm{Ha}^{-2}\,\partial\alpha/\partial\vartheta;\quad \delta\Phi = -im\alpha/(\eta s^2 c^2\,\mathrm{Ha}^2);\\ &P_0 = icm^{-1}(2 + s\partial/\partial s)\,c\delta\Phi - \delta P;\quad \Phi_0 = im^{-1}[(s\partial/\partial s - c^2\Delta)P_0 -\\ &\quad - (2 + s^2/c^2 + s\partial/\partial s)\,\delta P] - \delta\Phi. \end{aligned} \tag{14}$$

The problem is solved in principle by means of formulas (14) in that it has been reduced to the calculation of the derivatives of the given function $\alpha(\vartheta)$. We initially calculate δP, $\delta\Phi$, and this followed by the calculation of P_0 and Φ_0. All of the values are obtained at the r = 1 surface of the drop. In order to accomplish the transition of P_0 and Φ_0 into the interior of the drop the argument s should be replaced by ρ and the Hartmann boundary-layer factor exp [c Ha(r − 1)] should be assigned to δP and $\delta\Phi$. Repeated differentiation is used to calculate velocities (2).

The quantities δP, $\delta\Phi$, Φ_0, and P_0, just as the velocity in the main volume of the drop, are of the order of $1/\mathrm{Ha}^2$. In the Hartmann layer and at the surface the velocity is greater, i.e., on the order of 1/Ha.

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