SPACE EFFECTS IN MHD FLOW IN A CHANNEL WITH SECTIONED ELECTRODES

B. B. Volchek, L. R. Starominskii, and I. M. Tolmach

Magnitnaya Gidrodinamika, Vol. 3, No. 2, pp. 43–49, 1967

UDC 533.95:538.4

The flow energy indices are calculated for an MHD channel with sectioned electrodes of finite dimensions over a wide range of variation of the Hall parameter, relationship of electrode and insulator intervals and ratio of width of the flow to the electrode step. The results of the calculations are discussed.

In [1] an analytic description was found for the current density distribution in the flow of an electrically conducting fluid in a channel with an arbitrary number of sectioned electrodes of finite dimensions.

For a limited number of electrodes the problem reduces to a Hilbert boundary problem for a strip (non-periodic problem).

When there is an infinitely large number of electrodes and the pattern of current density distribution repeats itself from element to element (Fig. 1), the Hilbert problem is solved in terms of doubly periodic functions (periodic problem).

In the latter case the expression for the current density in relative units is as follows:

$$F(z) = \bar{j}_y + i\bar{j}_x = i\{a_1'[f(z)]^{\varphi/\pi} + a_2'[f(z)]^{\varphi/\pi - 1}\}, \quad (1)$$

where

$$f(z) = \frac{\sigma(z_1 - \lambda_1)\sigma_3(z_1 - \tau_1)}{\sigma(z_1)\sigma_3(z_1 - \lambda_1 - \tau_1)}. \quad (2)$$

Here a_1' and a_2' are real numbers, determined by the circuit joining the electrodes and the values of a series of dimensionless integrals H_1–H_4, T_1–T_4; σ and σ_3 are the Weierstrass sigma functions; $z_1 = (z/2\omega_1)$; $\lambda_1 = (\lambda/2\omega_1)$; $\tau_1 = (\tau/2\omega_1)$; $\varphi = \mathrm{arctg}\, 1/\beta$. All the symbols correspond to [1].

The current density distribution and energy indices are found for the following variants of the initial parameters of the periodic problem (Table 1).*

The displacement of oppositely situated electrodes in the limits of the element is $\tau_1 = 0$ in all variants. The optimal value of the load coefficient k was chosen in accordance with (20) in [1] for the values taken for the Hall parameter β and γ_1, the coefficient of inclination of the connectors to the y axis.

In what follows the current density distributions are calculated for the case when the electrode and insulator intervals have equal dimensions, and the ratio of electrode length to the step between electrodes is $\lambda_1 = 0.5$, and also for the case when the insulator is narrow compared with the electrode ($\lambda_1 = 0.9$). Calculations were made for a series of fixed values of β and ratio of width of flow to the step between electrodes $\omega_2/2\omega_1$. The variant $\lambda_1 = 0.9$ is important from the point of view of actually building a long-life channel with water cooled electrodes which occupy almost the whole wall in this case.

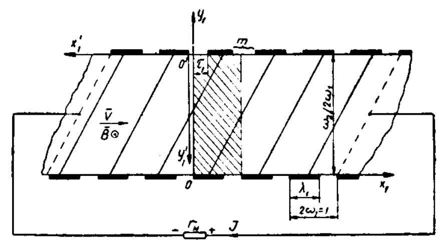

Fig. 1. Diagram of the MHD channel.

Using the properties of sigma-functions it can be shown that the current density distribution in the $x_1'y_1'$ coordinate system (Fig. 1) has exactly the same form as in the x_1y_1 system when the substitution $z_1' = x_1' + iy_1'$ is made instead of $z_1 = x_1 + iy_1$, where

$$z_1 = -z_1' + \lambda_1 + \tau_1 - 1 + i\frac{\omega_2}{2\omega_1}. \quad (3)$$

From this it follows that

$$\frac{\sigma(z_1 - \lambda_1)\cdot\sigma_3(z_1 - \tau_1)}{\sigma(z_1)\cdot\sigma_3(z_1 - \lambda_1 - \tau_1)} = \frac{\sigma(z_1' - \lambda_1)\cdot\sigma_3(z_1' - \tau_1)}{\sigma(z_1')\cdot\sigma_3(z_1' - \lambda_1 - \tau_1)} \quad (4)$$

when (3) is taken into account.

Thus the pattern of current density distribution posseses central symmetry and it suffices to calculate it in the lower half of the channel (the strip $z_1 = x_1 - x_1 + i(1/2)\cdot(\omega_2/2\omega_1)$, and to use the property indicated for the upper half.

At the same time the calculation of density for the lower half alone is made considerably easier in the case $\omega_2/2\omega_1 \geq 2$ (a case which can be realized in practice) since we then have, with an accuracy to 0.2%,

$$\frac{\sigma(z_1 - \lambda_1)\cdot\sigma_3(z_1 - \tau_1)}{\sigma(z_1)\cdot\sigma_3(z_1 - \tau_1 - \lambda_1)} \approx e^{-\frac{\pi^2}{3}\tau_1\lambda_1}\cdot\frac{\sin\pi(z_1 - \lambda_1)}{\sin\pi z_1}. \quad (5)$$

The simplification (5) follows from evaluating the terms of the series of products by which the sigma-functions are expressed.

On the basis of (5), for $\tau_1 = 0$ we have the following current density distribution on the imaginary axis $z_1 = iy_1$ (the distribution is valid within the limits $y_1 = 0$–$(\omega_2/4\omega_1)$):

$$\bar{j}_y + i\bar{j}_x = i\{a_1'[\cos\pi\lambda_1 + i\sin\pi\lambda_1\cdot\mathrm{cth}\,\pi y_1]^{\varphi/\pi} + a_2'[\cos\pi\lambda_1 + i\sin\pi\lambda_1\cdot\mathrm{cth}\,\pi y_1]^{\varphi/\pi - 1}\}. \quad (6)$$

*An arithmetical error was made in the numerical example of paper [1]. The corrected calculation is given in Table 2 of the present paper (variants 11–15).

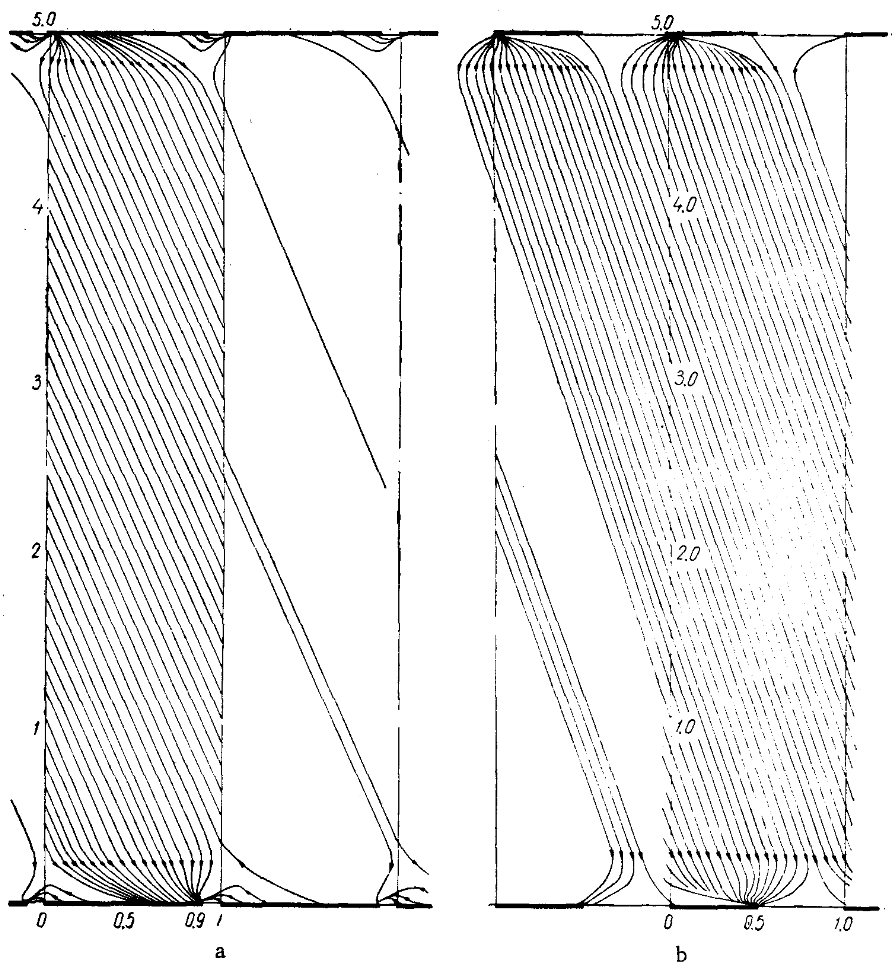

Fig. 2. a) Current density distribution in a channel for $\beta = 4$, $(\omega_2/2\omega_1) = 5$, $\lambda_1 = 0.9$, $k = 0.343$ (narrow insulators). b) Current density distribution in a channel for $\beta = 4$, $(\omega_2/2\omega_1) = 5$, $\lambda_1 = 0.5$, $k = 0.343$ (equal insulators and electrodes).

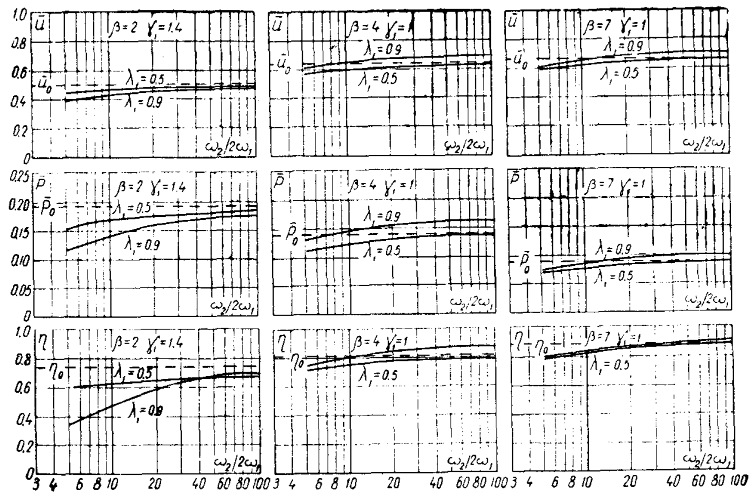

Fig. 3. Flow energy indices—voltage $\overline{U}$, load power $\overline{P}$, and efficiency as functions of the ratio of width of flow to section step. $\overline{U}_0$, $\overline{P}_0$, η_0 are the corresponding values of these quantities given by the Montardi theory.

For a change of y_1 from $\omega_2/4\omega_1$ to $\omega_2/2\omega_1$ the expression for the density on the imaginary axis, taking (3), (4), and (5) into account, is given by

$$\bar{j}_y+i\bar{j}_x=$$

$$=i\left\{a_1'\left[\frac{1}{\cos\pi\lambda_1+i\sin\pi\lambda_1\cdot\operatorname{cth}\pi\left(y_1-\frac{\omega_2}{2\omega_1}\right)}\right]^{\varphi/\pi}+\right.$$

$$\left.+a_2'\left[\frac{1}{\cos\pi\lambda_1+i\sin\pi\lambda_1\cdot\operatorname{cth}\pi\left(y_1-\frac{\omega_2}{2\omega_1}\right)}\right]^{\varphi/\pi-1}\right\}. \quad (6a)$$

On the real axis, taking (5) into account, it is

$$\bar{j}_y+i\bar{j}_x=i\left\{a_1'\left[\frac{\sin\pi(x_1-\lambda_1)}{\sin\pi x_1}\right]^{\varphi/\pi}+\right.$$

$$\left.+a_2'\left[\frac{\sin\pi(x_1-\lambda_1)}{\sin\pi x_1}\right]^{\varphi/\pi-1}\right\}. \quad (7)$$

Expressions (6), (6a), and (7) facilitate finding the integrals T_1-T_4 and H_1-H_4.

For the corresponding variants of Table 1 calculated values are given in Table 2 for the integrals mentioned above, the constants a_1' and a_2' and the energy indices:

voltage on the load (unit voltage is equal to VBl)

$$\bar{u}=-a_1'H_3-a_2'H_4; \quad (8)$$

useful power (unit of power is equal to $\gamma(VB)^2\omega_2\Delta l$)

$$\bar{P}=-\frac{2\omega_1}{\omega_2}(a_1'H_3+a_2'H_4)\times$$

$$\times[a_1'(T_3-mH_1)+a_2'(T_4-mH_2)]; \quad (9)$$

total electric power

$$\bar{P}_e=\frac{2\omega_1}{\omega_2}\left\{a_1'T_1+a_2'T_2+\int_0^1\left[\int_0^{x_1}(\bar{j}_{x_0}-\bar{j}_{xH})\,dx_1\right]dx_1\right\}. \quad (10)$$

Table 1

Nos. of Variants	$\frac{\omega_2}{2\omega_1}$	λ_1	β	γ_1	m	κ
1—5	5, 10, 20, 40, 100	0.9	7	1	5, 10, 20, 40, 100	0.2
6—10	the same	0.5			the same	
11—15	5, 10, 30, 40, 100	0.9	4	1	5, 10, 20, 40, 100	0.343
16—20	5, 10, 20, 40, 100	0.5			the same	
21—25	the same	0.9	2	1.4	7, 14, 28, 56, 140	0.77
26—30	the same	0.5			the same	

Note: the number m corresponds to the chosen inclination of the connectors.

The same table gives the corresponding values of these quantities calculated for the Montardi configuration, infinitesimally small electrodes and insulators and a current which is assumed to be strictly uniform entering (and leaving) the edges of the channel:

$$\bar{U}_0=\frac{\beta+\gamma_1}{1+\gamma_1^2+k(1+\beta^2)}; \quad (11)$$

$$\bar{P}_0=\frac{k(\beta+\gamma_1)^2}{[1+\gamma_1^2+k(1+\beta^2)]^2}; \quad (12)$$

$$\bar{P}_{e0}=\frac{1+k}{1+\gamma_1^2+k(1+\beta^2)}; \quad (13)$$

$$\eta_0=\frac{\bar{P}_0}{\bar{P}_{e0}}. \quad (14)$$

It is clear from the table that for large values of the ratio of width of flow to step of the section the

Table 2

$T_1\cdot(-1)$	T_2	T_3	T_4	a_1'	$a_2'\cdot(-1)$	U	P	$P_e\cdot(-1)$	η
$H_1=-0.127$; $H_2=0.447$; $H_3=0.0976$; $H_4=1.997$; $\overline{U}_0=0.667$; $\overline{P}_0=0.0889$; $\overline{P}_{e0}=0.1$; $\eta_0=0.889$									
0.6227	4.364	4.944	−4.658	−0.252	0.293	0.6096	0.0748	0.0936	0.799
1.2597	6.5094	9.904	−9.173	−0.27	0.318	0.6614	0.0872	0.1038	0.84
2.5327	10.794	19.834	−18.208	−0.278	0.33	0.6861	0.0942	0.1090	0.865
5.0827	19.56	39.644	−36.258	−0.277	0.331	0.700	0.0975	0.1095	0.888
12.7227	45.104	99.144	−90.458	−0.285	0.341	0.707	0.1003	0.1108	0.907
$H_1=-0.0705$; $H_2=1.000$; $H_3=0.4981$; $H_4=7.063$; $\overline{U}_0=0.667$; $\overline{P}_0=0.0889$; $\overline{P}_{e0}=0.1$; $\eta_0=0.889$									
0.3512	11.6	4.956	0.818	0.0433	0.08725	0.5944	0.0706	0.0892	0.792
0.7042	16.59	9.946	1.167	0.0418	0.0921	0.6302	0.0793	0.977	0.81
1.411	26.566	19.921	1.866	0.041	0.0947	0.6486	0.0842	0.0982	0.855
2.821	46.5	39.86	3.268	0.0406	0.0959	0.6573	0.0863	0.099	0.87
7.071	106.36	99.706	7.458	0.040	0.09725	0.667	0.0887	0.1	0.887
$H_1=-0.216$; $H_2=0.552$; $H_3=0.1023$; $H_4=1.156$; $\overline{U}_0=0.639$; $\overline{P}_0=0.140$; $\overline{P}_{e0}=0.1715$; $\eta_0=0.815$									
1.068	3.866	4.837	−3.99	−0.39	0.4982	0.615	0.1303	0.1742	0.748
2.1615	6.426	9.716	−8.285	−0.4267	0.529	0.6553	0.1468	0.1817	0.806
4.348	11.536	19.477	−16.7	−0.445	0.546	0.6766	0.1553	0.1845	0.841
8.718	21.766	39.037	−34.09	−0.4517	0.551	0.6822	0.1612	0.1865	0.865
21.86	52.56	97.53	−85.55	−0.453	0.553	0.6844	0.1615	0.1869	0.865
$H_1=-0.1177$; $H_2=0.966$; $H_3=0.4866$; $H_4=3.996$; $\overline{U}_0=0.639$; $\overline{P}_0=0.140$; $\overline{P}_{e0}=0.1715$; $\eta_0=0.815$									
0.6076	8.405	4.937	1.03	0.0731	0.152	0.572	0.112	0.1555	0.72
1.218	13.37	9.902	1.638	0.071	0.159	0.602	0.124	0.1636	0.758
2.43	23.32	19.85	2.852	0.070	0.163	0.618	0.131	0.169	0.775
4.874	43.12	39.65	5.287	0.0696	0.165	0.6262	0.135	0.171	0.79
12.19	102.67	99.2	12.587	0.069	0.166	0.6304	0.136	0.1724	0.79
$H_1=-0.396$; $H_2=0.661$; $H_3=0.1027$; $H_4=0.683$; $\overline{U}_0=0.5$; $\overline{P}_0=0.192$; $\overline{P}_{e0}=0.26$; $\eta_0=0.739$									
1.992	5.368	4.592	−3.245	−0.35	0.521	0.392	0.117	0.346	0.338
4.017	8.768	9.167	−6.945	−0.405	0.573	0.4326	0.1422	0.296	0.481
8.067	15.568	18.322	−14.345	−0.435	0.603	0.4567	0.1595	0.272	0.587
16.182	29.168	36.632	−29.145	−0.45	0.618	0.468	0.1692	0.257	0.654
40.492	69.968	91.592	−73.545	−0.458	0.627	0.474	0.1742	0.2508	0.695
$H_1=-0.222$; $H_2=0.891$; $H_3=0.4956$; $H_4=1.993$; $\overline{U}_0=0.5$; $\overline{P}_0=0.192$; $\overline{P}_{e0}=0.26$; $\eta_0=0.739$									
1.135	6.52	4.789	1.553	0.0905	0.25	0.453	0.158	0.265	0.596
2.27	11.38	9.659	2.713	0.0866	0.258	0.471	0.1705	0.271	0.629
4.535	21.12	19.39	5.033	0.084	0.261	0.479	0.1764	0.273	0.646
9.085	40.57	38.889	9.678	0.0847	0.263	0.483	0.1804	0.2745	0.6575
22.735	98.92	97.29	23.58	0.0827	0.266	0.489	0.184	0.2773	0.664

quantities $\bar{U}$, $\bar{P}$, $\bar{P}_e$, η tend to certain limiting values depending on λ_1 and β and do not coincide with the values $\bar{U}_0$, $\bar{P}_0$, $\bar{P}_{e0}$, η_0 (although the discrepancy is not great for the combinations of parameters under consideration). This is explained by the difference in formulation of the Montardi problem and the present problem. Even for very fine sectioning the presence of a large number of peaks in the current distribution along the edges with electrodes introduces a total contribution to the energy indices which causes the discrepancy we have noted.

Patterns of the current density distribution are given in Fig. 2 for $\beta = 4$, $(\omega_2/2\omega_1) = 5$, $k = 0.343$, and for the values $\lambda_1 = 0.9$ (Fig. 2a) and 0.5 (Fig. 2b).

It is clear that in the first case (narrow insulators) reverse current flow arises along the insulating intervals.

In the second case (equal electrodes and insulators) such flow does not arise.

In the central part of the channel the density is very uniform indeed. Distortions arise at a distance of the order of the section step. The central symmetry of the density distribution mentioned above is clear from the figures.

Figure 3 gives $\bar{U}$, $\bar{P}$ and η as functions of the ratio $\omega_2/2\omega_1$, for all variants of Table 1. The values in Fig. 3 with bars $\bar{U}_0$, $\bar{P}_0$ and η_0, are those corresponding to the Montardi problem. For values of the ratio $\omega_2/2\omega_1$ which are not large (less than 10) the energy indices obtained in all cases are lower than U_0, P_0, η_0, and increase as this ratio increases.

The behavior of the curves $\bar{U}$, $\bar{P}$ and η has a marked dependence on the Hall parameter β. Thus for not very large values ($\beta = 2$) lower indices are obtained for a channel with narrow insulators ($\lambda_1 = 0.9$) than for $\lambda_1 = 0.5$. As β increases the curves for $\lambda_1 = 0.9$ go higher than the corresponding curves for $\lambda_1 = 0.5$.

For large values of β (4–7) in the case when the value of the ratio $\omega_2/2\omega_1 \geq 10$ the magnitudes of the efficiency and other energy indices are close to the values given by the Montardi theory, and may even exceed them as this ratio increases.

In making concrete flow calculations from the one-dimensional equations and using formulas (11)–(14) estimates from Fig. 3 are made more accurate by multiplying the power by the corresponding coefficient $\bar{P}/\bar{P}_0$, to take the effect of two dimensions into account.

REFERENCE

1. I. M. Tolmach and N. N. Yasnitskaya, Izv. AN SSSR, Energetika i transport, 5, 91, 1965.

4 October 1966