*Magnetohydrodynamics, Vol. 31, Nos. 1-2, 1995*

# MAGNETIC FIELD GENERATION BY THE AXISYMMETRIC CONDUCTING FLUID FLOW IN A SPHERICAL CAVITY OF A STATIONARY CONDUCTOR. 2

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*The magnetic field generation by a distributed flow having a polynomial stream function in a uniformly conducting medium is considered. It is found that the simplest meridional flow in the shape of two contacting ring vortices generates a steady magnetic field. The critical $Re_m$ value and the field pattern are calculated. If the surface velocities are directed toward the equator, a dipole-type field is generated; if they are directed toward the poles, the field is of the quadrupole type.*

The present report is a continuation of [1], in which a spatially unbounded, stationary conductor containing a unit radius (R = 1) cavity filled with a fluid, whose electrical conductivity $\sigma$ is the same as that of the conductor, was considered. Three axially symmetric flow types of that fluid were investigated for a spontaneous magnetic field generation. The background of the problem was presented in [1]. For two flow types (a single vortex ring with an azimuthal rotation and a pair of noncontacting ring vortices without any azimuthal motion) the existence of generation was established. For the third, which mathematically is the simplest flow type (a pair of contacting vortex rings without azimuthal motion) it was found that the accuracy of the calculation was insufficient to make a definite judgement. In the present investigation, the study of the third flow type was continued. The calculation accuracy is improved and the existence of generation is confirmed.

**Formulation of the Problem.** We consider an axially symmetric flow of an incompressible fluid v in the ($r \leq R = 1$) space, which in the spherical coordinates r, $\theta$, and $\varphi$ can be specified by the equation

$$v_r/v_{max} = r\,(1 - r^2)\,(3\cos^2\theta - 1); \quad v_\theta/v_{max} = r\,(5r^2 - 3)\sin 2\theta/2, \quad v_\varphi = 0. \tag{1}$$

The flow pattern (see Fig. 1 in [1]) is mathematically the simplest flow with a polynomial (in the cylindrical coordinates) stream function and a topology of two contacting vortex rings. We wish to find the value of $Re_m = \mu_\theta \sigma v_{max} R$ at which a constant non-axisymmetric magnetic field **B** is induced. Due to the axial symmetry of the flow, the spherical field components exponentially depend on the azimuth $\varphi$:

$$B_r,\ B_\theta,\ B_\varphi \sim \exp(im\varphi). \tag{2}$$

Next we investigate the case m = 1. In principle, the problem formulation does not differ from that described in [1]. The field is represented in terms of the poloidal and toroidal components which are expanded into spherical angular functions $P^m{}_l(\cos\theta)\exp(im\varphi)$. The induction equation gives rise to an infinite system of radial differential equations which is cut off at some maximum $l = l_{max}$ and is then numerically integrated.

The distinguishing feature of the considered phenomenon is a numerically large value of the critical $Re_m \sim 10^3$. Because of that the calculations require $l_{max} \sim 100$ and several hundreds of radial integration steps. The large $Re_m$ renders the traditional integration schemes unstable and requires that relaxation methods be applied [2] which reduce the integration to the solution of an extremely large system of linear algebraic difference equations.

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Translated from Magnitnaya Gidrodinamika, Vol. 31, Nos. 1-2, pp. 39-43, January-June, 1995. Original article submitted June 17, 1994.

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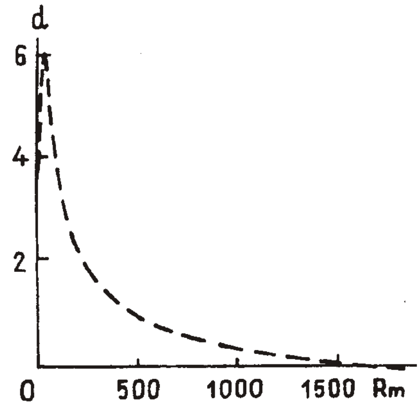


Fig. 1. Exciting current $d(Re_m)$ at the cavity surface which is required to support the normalized field at the cavity center (limiting curve).

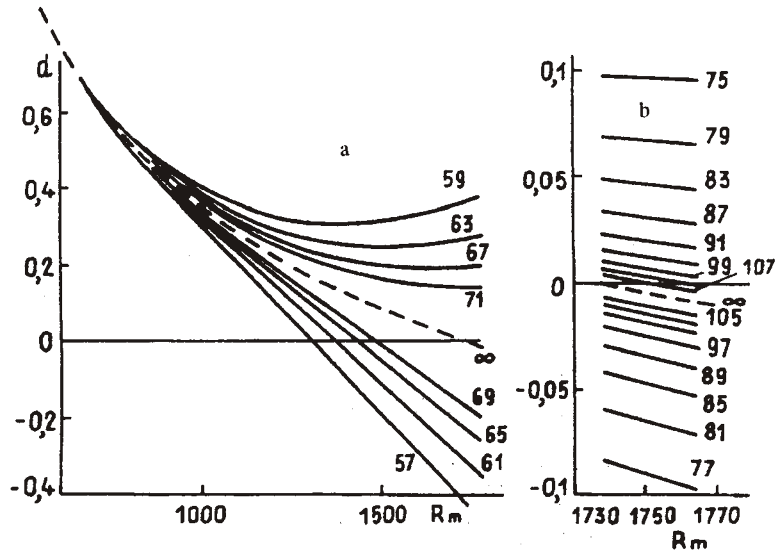


Fig. 2. The convergence of the computed $d(Re_m)$ curves to the limiting one as $l_{max}$ is increased. The $l_{max}$ values are indicated on the curves.

As in [1], here we also will not replicate all the equations. As in [1], to evaluate the critical $Re_m$, an intermediate $d(Re_m)$ quantity is introduced. It represents the "electric current which must be passed across the cavity surface in order to maintain the normalized field at its center." The critical $Re_m$ is determined as the root of the $d(Re_m) = 0$ equation, which signifies that in self-excitation no maintenance current is required.

**Refinement of the Calculation.** The present investigation differs from [1] above all by the significant increase in $l_{max}$. In [1], this was prevented by overloading the computer memory with 1 megabyte of intermediate data, in particular, with the "inverse formulation matrix." In the present investigation, the matrix structure was changed to allow its retention in the RAM memory in parts which can be removed from the memory according to the use in evaluating $d(Re_m)$. This was found to be possible, since $d(Re_m)$ enters into the equations linearly and its refinement does not require that relaxation iterations be performed. This reduces only the round-off errors which, as demonstrated in [1], in a doubled accuracy computation are insignificant. The entire inverse formulation matrix is necessary in order to construct the field map. In this case, the hard disk memory is used. The computation was performed using a Gateway 2000 (486/33) which is a somewhat smaller computer than that used in [1].

The second difference concerns the order of computation. Due to the computation length, there is no practical possibility of selecting such large $l_{max}$ and such small radial integration step h as to make it possible to simply neglect the break-off and integration errors.

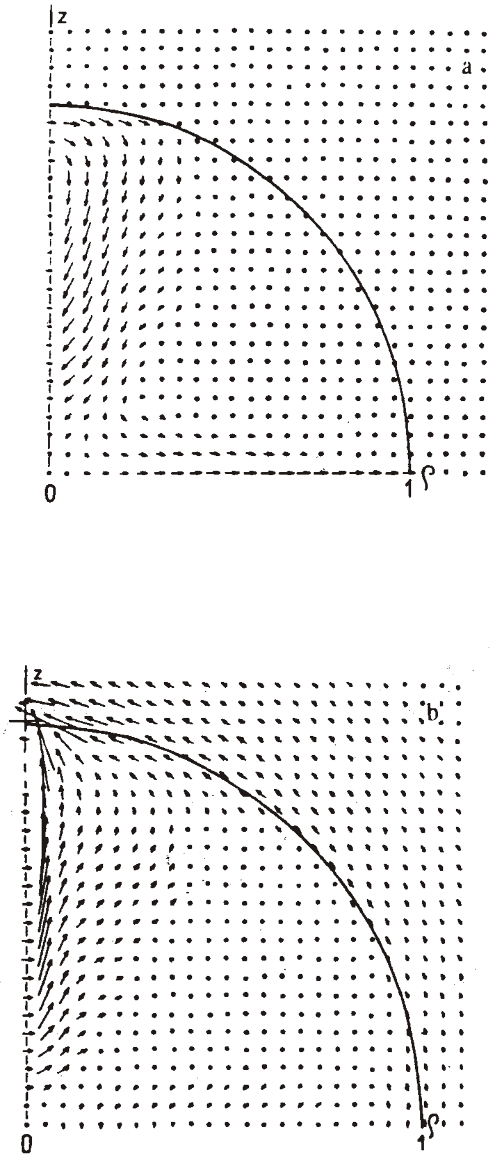


Fig. 3. Pattern of the meridional magnetic field components for a flow from the poles to the equator (a) and from the equator to the poles (b). The direction and length of the arrows indicate the direction and numerical value of the field.

In [1], the $d(Re_m, l_{max}, h) = 0$ equation was solved for a series of $l_{max}$ and h, and the $Re_m$ root itself was exposed by a graphical extrapolation to the $l_{max} \to \infty$, $h \to 0$ limit. In the present investigation, the $d(Re_m)$ is refined first, while the critical $Re_m$ is evaluated as the root of the limiting curve $d(Re_m) = 0$ (Fig. 1).

In order to construct this curve, one first goes to the $h \to 0$ limit. For a fixed $l_{max}$, the radius is divided into n equal integration steps, and $d(Re_m)$ are calculated for three n values. Instead of a graphical extrapolation, the limiting $d = d(n = \infty)$ value for $n = \infty$ is calculated, excluding the corrections $a$ and b, from the three equations

$$d(n) = d + a/n^2 + b/n^4; \qquad n = n_1, n_2, n_3. \tag{3}$$

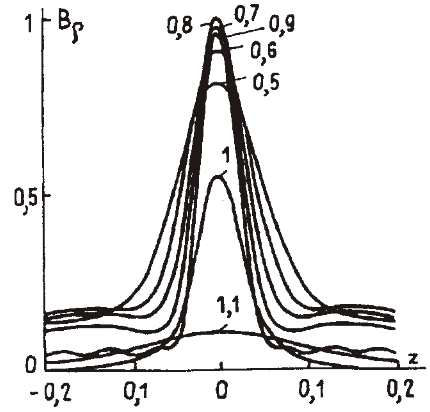


Fig. 4. Structure of the equatorial layer in the cylindrical coordinates (radial field at the various radii $\rho$ as a function of the axial distance z). The $\rho$ values are indicated on the curves.

(the relaxation method of [2] employs the second-order smallness approximation, where the leading correction is quadratic in $h \sim 1/n$ and there are no odd corrections because the difference equations do not change with the sign of h, i.e., with the integration direction).

With increasing $l_{max}$ the $d(Re_m)$ curves generate a cluster which slowly converges toward the limiting curve (Fig. 2). The following values were selected to generate Fig. 2a: $n_1 = 100$, $n_2 = 150$, $n_3 = 200$ (in the left-hand portion) and $n_1 = 150$, $n_2 = 200$, and $n_3 = 300$ (for some points in its right-hand portion). For Fig. 2b the values were $n_1 = 200$, $n_2 = 300$, and $n_3 = 600$.

The $l_{max}$ break-off parameter was increased until the existence of the $d(Re_m, l_{max}) = 0$ root became obvious. The limiting curve was drawn from the 1-MB limit of the averaged weighted quantities

$$\bar{d}_l = 1/16\, d_{l-4} + 1/4\, d_{l-2} + 3/8\, d_l + 1/4\, d_{l+2} + 1/16\, d_{l+4} \tag{4}$$

from five neighboring $l_{max}$ and intersected the axis at

$$Rm = 1734{,}95 \pm 0{,}01. \tag{5}$$

(The positive $Re_m$ values in the employed coordinate system correspond to a motion on the cavity surface directed toward the equator.)

It is evident from the map of the meridional excited magnetic field components (Fig. 3a) that a dipole-type field is generated perpendicular to the axis of symmetry. Qualitatively the field resembles that produced by a pair of noncontacting vortices (see [1], Fig. 6), only emerging outwardly to a lesser degree. The field is strongly squeezed toward the axis and to the equatorial plane. A reliable mapping of a thin equatorial layer (Fig. 4) requires many spherical functions. The small near-surface field oscillations evidently can be explained by the imperfection of the break-off procedure, since the oscillation period corresponds to the neglected spherical function.

An analogous computation for the quadrupole-type field yields

$$Rm = -\ 1734{,}9 \pm 0{,}5.$$

The negative sign means that the solution is established for a reverse flow direction when the fluid on the cavity surface is flowing toward the poles. The equality of the absolute values requires the Proctor theorem [3, 4]. The agreement of the computed values is an argument for a complete correctness of the calculation.

Thus, the flow specified by Eq. (1) can be considered to be the simplest purely meridional axisymmetric flow producing a magnetic field.

## REFERENCES


1. A. Gailitis, "Magnetic field generation by an axially symmetric flow of a conducting fluid in a spherical cavity of a stationary conductor," Magn. Gidrodin., No. 2, 3-14 (1993).
2. W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, Numerical Recipes, CUP, Cambridge (1986).
3. M. R. E. Proctor, "On the eigenvalues of kinematic $\alpha$-effect dynamos," Astr. Nachr., **298**, 19-25 (1977).
4. M. R. E. Proctor, "The role of mean circulation in parity selection by planetary magnetic fields," Geophys. Astrophys. Fluid Dyn., **8**, 311-324 (1977).