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BUOYANCY DRIVEN FLOW AND ITS STABILITY IN A HORIZONTAL RECTANGULAR CHANNEL WITH AN ARBITRARY ORIENTED TRANSVERSAL MAGNETIC FIELD

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An MHD flow is considered which is relevant to horizontal Bridgman technique for crystal growth from a melt. In the unidirectional parallel flow approximation an analytical solution is found accounting for the finite rectangular cross section of the channel in the case of a vertical magnetic field. Numerical pseudo-spectral solutions are used in the cases of arbitrary magnetic field and gravity vector orientations. The vertical magnetic field (parallel to the gravity) is found to be he most effective to damp the flow, however, complicated flow profiles with "overvelocities" in the corners are typical in the case of a finite cross-section channel. The temperature distribution is shown to be dependent on the flow profile. The linear stability of the flow is investigated by use of the Chebyshev pseudospectral method. For the case of an infinite width channel the transversal rolls instability is investigated, and for the finite cross-section channel the longitudinal rolls instability is considered. The critical Gr number values are computed in the dependence of the Ha number and the wave number or the aspect ratio in the case of finite section.

Introduction

The recent interest in MHD flows in horizontal channels is motivated by their relevance to the horizontal Bridgman technique for semiconductor crystal growth [1, 2]. The specific property of these flows is that they are generated by a rotational buoyancy force due to a nearly uniform horizontal temperature gradient, and a transversal magnetic field creates additional rotational electromagnetic force in the fluid, which, on general, damps the initial flow and enhances its stability. However, the velocity profile in a channel of finite cross section is significantly different in the presence of the second vortical force, which can lead to highly sheared flows with "overvelocities" relative to the central core flow. At moderate Hartmann numbers the stability of such flows can be of practical importance for crystal growth applications.

Statement of the Problem

A long horizontal rectangular cavity is heated at one end and cooled at the opposite in the x-coordinate direction. The gravity vector $\mathbf{g} = g_y\mathbf{e}_y + g_g\mathbf{e}_g$ can be oriented arbitrarily in the transversal plane (y, z) thus permitting one to simulate a fixed rotation of the channel, e.g., a diagonally oriented channel. The horizontal cavity of a rectangular cross section is assumed to be sufficiently long to motivate the unidirectional flow approximation far from the end walls. But different from many previous investigations of similar buoyancy driven flows (the book [3] contains an extensive reference list on the subject), we will keep the lateral dependence of y-coordinate, in addition to vertical z-coordinate, for the finite width channel.

The governing equations are expressed in a nondimensional form by use the following scales: the depth h of the channel for the coordinates, velocity $V_0 = \nu/h$ (ν is kinematic viscosity), time h/V_0, pressure ρV_0^2 (ρ is the fluid density), temperature $T_0 - \Delta T \cdot h/l$ according to the uniform gradient along the length l of the cavity, and for magnetic field — the magnitude of the external uniform field B_0. Then the Reynolds number $Re = V_0 h/\nu = 1$, the Grashof number $Gr = \beta g(\Delta T/l)h^4/\nu^2$ (β is coefficient of thermal expansion, g — gravity), the Hartmann number $Ha = B_0 h(\sigma/\rho\nu)^{1/2}$, the Peclet and

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Prandtl numbers $Pe = Re \cdot Pr = \nu/\alpha$ (α — thermal diffusivity), and the magnetic Reynolds number $Rm = \nu\mu\sigma$ (σ — electrical conductivity, μ — magnetic permeability). The parallel basic flow approximation means that the velocity can be represented as $\mathbf{V} = U(y, z)\mathbf{e}_x + \mathbf{v}(t, x, y, z)$, $\mathbf{v} = \{u, v, w\}$ for Cartesian components of the time dependent perturbation velocity in the stability analysis. Similarly the nondimensional temperature field can be represented as $T = x + T_1(y, z) + \Theta(t, x, y, z)$. Since the Rm number is typically very small, the nondimensional magnetic field can be approximated by the first order terms in Rm expansion: $\mathbf{B} = B_{0z}\mathbf{e}_z + B_{0y}\mathbf{e}_y + Rm(B_{1x}(y, z)\mathbf{e}_x + \mathbf{b}(t, x, y, z))$.

Substituting these representations in the MHD equations we obtain for the stationary variables the following problem:

$$\Delta U + Ha^2(B_{0y}\partial_y B_{1x} + B_{0z}\partial_z B_{1x}) = Gr(g_z z + g_y y), \tag{1}$$

$$\Delta B_{1x} + B_{0y}\partial_y U + B_{0z}\partial_z U = 0, \tag{2}$$

$$PeU = \Delta T_1, \tag{3}$$

where $\Delta = \partial_{yy} + \partial_{zz}$.

Equation (1) is obtained by integrating the y- and z-components of the curl of momentum equations respectively by z and y. Notice, also, that the motion induced electric current can be expressed from $\text{curl}\,\mathbf{B} = \mathbf{j}$, thus the nondimensional stationary electric current components are

$$j_y = \partial_z B_{1x}, \quad j_z = -\partial_y B_{1x}, \tag{4}$$

and the electric current lines are given by $B_{1x} = \text{const}$.

The respective boundary conditions on the walls $z = \pm 1/2$ and $y = \pm y_0$ are no-slip for velocity:

$$U = 0, \tag{5}$$

for the heat insulated walls:

$$\partial_n T = 0, \tag{6}$$

and for the electrically insulated walls the electric current normal component is zero, which with (4) means that

$$B_{1x} = const_0 = 0. \tag{7}$$

Solutions for the Stationary Problem

In some cases with special symmetries it is possible to construct analytical solutions, i.e., when the gravity is purely vertical, $g_y = 0$. The simplest is a nonmagnetic case with $Ha = 0$, but for the finite width of the channel. The solution of (1), (5) is

$$U = \mathrm{Gr}\left(\frac{z^3}{6} - \frac{z}{24} + \sum_{n=1}^{\infty} a_n \mathrm{ch}\, ky \cdot \sin kz\right)$$
$$\text{where } k = 2\pi n, \; a_n = (-1)^{n+1}/(4\pi^3 n^3 \mathrm{ch} k y_0). \tag{8}$$

One can observe that in the central part of the channel, $y \to 0$, there is an exponential convergence of the series (8) to the cubic profile of the infinite width channel flow [3], and at the walls $y \to \pm y_0$ there is n^{-3}-like convergence of the series.

For the case of vertical gravity and vertical magnetic field ($B_y = 0$) the solution to (1), (2), (5), (7) can also be expressed in an analytical form

$$U = \mathrm{Gr}\,\mathrm{Ha}^{-2}(sh(\mathrm{Ha}\, z)/(2sh(\mathrm{Ha}/2)) - z) + \sum_{n=1}^{\infty}\begin{bmatrix} \mathrm{Ha}(a_n sh(\lambda y)\cdot \sin ly - b_n ch(\lambda y)\cdot \cos ly)\sin kz \\ +(c_n \kappa^{-1}(\mu^2 - \kappa^2) sh(\kappa z) + d_n \chi^{-1}(\mu^2 - \chi^2) sh(\chi z)\cos \mu y \end{bmatrix}, \tag{9}$$

where $k = 2\pi n$, $l = \left(k/2(\sqrt{k^2 + \mathrm{Ha}^2} - k)\right)^{1/2}$, $\lambda = kHa/(2l)$, $\mu = \pi y_0^{-1}(n - \frac{1}{2})$,

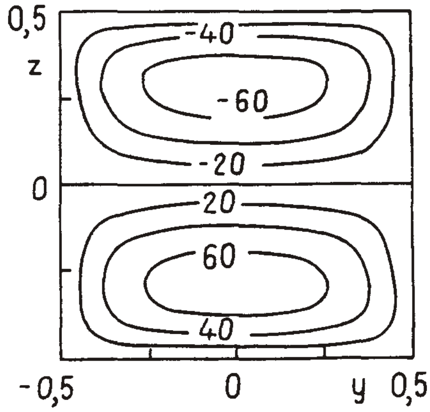

Fig. 1. Constant velocity lines.
Gr = 10,000, Ha = 0, g_z = 1.

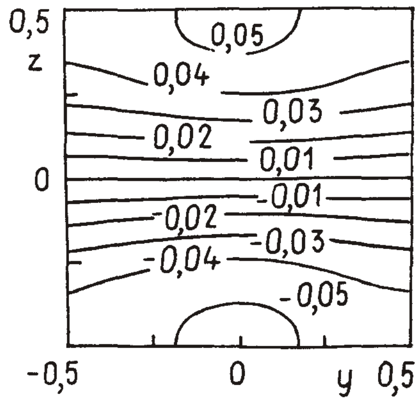

Fig. 2. Temperature T_1 lines.
Gr = 10,000, Ha = 0, g_z = 1.

$$\kappa = \left(\tfrac{1}{2}Ha^2 + \mu^2 - Ha\sqrt{\tfrac{1}{4}Ha^2 + \mu^2}\right)^{1/2}, \quad \chi = \left(\tfrac{1}{2}Ha^2 + \mu^2 - Ha\sqrt{\tfrac{1}{4}Ha^2 + \mu^2}\right)^{1/2};$$

the expressions for a_n, b_n, c_n, d_n determined by the boundary conditions are skipped to save space.

The solution (9) satisfies the following properties: the nonmagnetic solution (8) is recovered when Ha → 0. In the other asymptotic limit when Ha → ∞ the solution presents typical Hartmann layer behavior in thin layers of thickness $\Delta z \sim Ha^{-1}$ at the walls $z = \pm 1/2$ normal to the magnetic field, and a typical shear layer behavior $\Delta y \sim Ha^{-1/2}$ at the walls $y = \pm y_0$

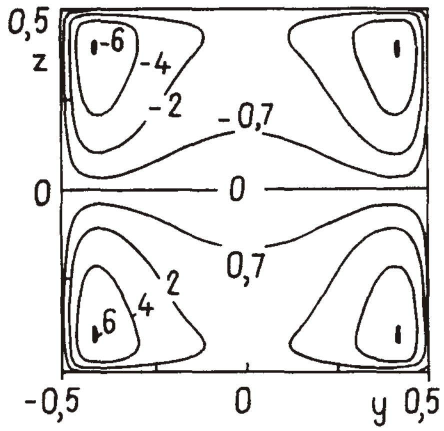

Fig. 3. Constant velocity lines. Gr = 10,000, Ha = 50, g_z = 1.

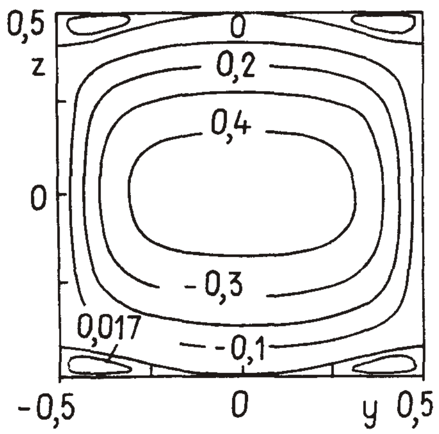

Fig. 4. Electric current lines. Gr = 10,000, Ha = 50, g_z = 1.

along the magnetic field. This can be deduced from the exponential terms of (9) where χ, κ ~ Ha and λ ~ $Ha^{1/2}$. For the case Ha → ∞ it is possible to also find the core flow behavior: since the coefficients a_n, b_n, c_n, d_n decrease there as $e^{-\sqrt{Ha}}$, we have the y-independent solution [1], [2] in the core.

The solutions (8) and (9) (with the respective expression for B_{1x}) are used as reference and test solutions for the more general pseudospectral solutions of the problem (1)-(3), (5)-(7) for arbitrary orientations of the imposed magnetic field and gravity vectors in the transversal plane.

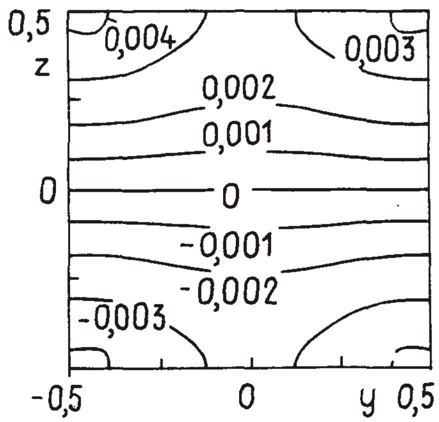

Fig. 5. Temperature T_1 lines. Gr = 10,000, Ha = 50, H_z = 1, g_z = 1.

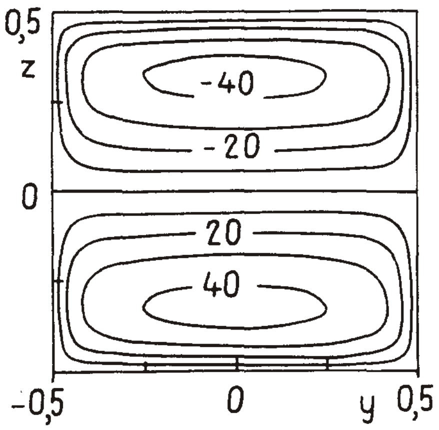

Fig. 6. Constant velocity lines. Gr = 10,000, Ha = 50, B_y = 1, g_z = 1.

The pseudospectral solutions were constructed, first, transforming the coordinates to

$$\bar{y} = y/y_0\,, \quad \bar{z} = 2z$$

with the purpose of applying Chebyshev polynomial $T_n(x)$ expansions on the interval $[-1, 1]$. (To simplify the notation we will omit the bars in the following.) The velocity U is expanded in the spectral functions satisfying the zero boundary conditions (5):

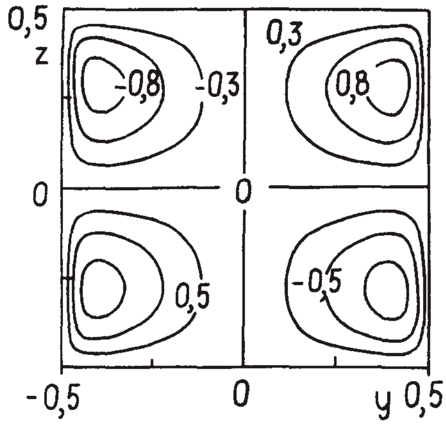

Fig. 7. Electric current lines. Gr = 10,000, Ha = 50, B_y = 1, g_z = 1.

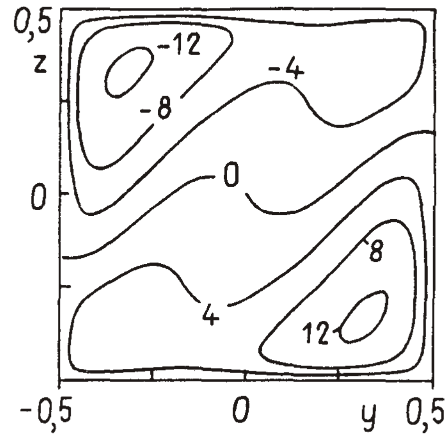

Fig. 8. Constant velocity lines, Gr = 10,000, Ha = 50, B_y = B_z = 71, g_y = g_z = 0.71.

$$U(y,z)=\sum_{m=1}^{M}\sum_{n=1}^{N}U_{mn}F_0(m,y)F_0(n,z)\,, \tag{10}$$

$$\text{where } F_0(k,x)=\begin{cases}T_{k+1}(x)-x\,; & k=2,4,\ldots \text{ even}\\ \mathrm{T}_{k+1}(x)-1\,; & k=1,3,\ldots \text{ odd}\end{cases}\,. \tag{11}$$

A similar expansion is also used for the magnetic field B_{1x}:

$$B_{1x}(y,z)=\sum_{m=1}^{M}\sum_{n=1}^{N}B_{mn}F_0(m,y)F_0(n,z)\,. \tag{12}$$

The temperature

$$T_1(y,z)=\sum_{m=0}^{M}\sum_{n=0}^{N}T_{mn}F_1(m,y)F_1(n,z) \tag{13}$$

is expanded in the spectral functions satisfying the conditions (6):

$$F_1(k,x)=\begin{cases}T_{k+2}(x)-(k+2)^2/4\;T_2(x)\;;\;k=2,4,\dots\text{ even}\\ T_{k+2}(x)-(k+2)^2T_1(x)\;;\;k=1,3,\dots\text{ odd}\end{cases}\,. \tag{14}$$

These expansions are substituted in Eqs. (1), (2), (3) and discretized on the Gauss–Chebyshev grid (zeros of $T_N(x)$) for the same total number of points N = M = 24 in y and z directions respectively. The derivatives of Chebyshev polynomials are computed according to the recommendations in [4]. The resulting set of algebraic equations is solved by the double precision routine DGESV from the LAPACK. The results were checked by varying the number of points (terms in the expansions) and against the analytical solutions (8), (9) — the agreement is found to be good.

The computed solutions for Gr = 100000, Pe = 0.01 are represented graphically. Figure 1 presents the computed constant velocity lines in a square channel for the nonmagnetic case [corresponding to the solution (8)]. The flow is symmetric in y and antisymmetric in z directions, with a small region of central flow corresponding to the previous y-independent solutions [3], [5], [6]. The flow in a finite channel perturbs the stationary temperature distribution over the cross-section as shown in Fig. 2.

The velocity profile in the finite width channel for the magnetic case Ha = 50 (Fig. 3) is significantly different from the y-independent solution in the infinite width flow. The "overvelocities" in the vicinity of corners are evidently explained by the electric current lines (see Fig. 4) being directed along the magnetic field in the side wall regions and, thus, no significant braking of the buoyancy driven flow in this region, although there is a strong damping in the main core flow which is nearly y-independent and corresponds to the solutions in [1], [8]. For a larger with channel the corner jets start to develop at even smaller Ha numbers, e.g., for y_0 = 1.5 — beginning with Ha = 5. The changes in the flow profile also strongly affect the temperature distribution over the channel — more heated regions are moved to the corners (Fig. 5) if compared to the nonmagnetic case (Fig. 2) with a top wall heated in the center. The effect of a horizontal magnetic field B_{0y} is much less if compared to the vertical field case, see Figs. 6 and 3.

In the horizontal field case the core flow damping is significantly lower (like Ha^{-1}) instead of Ha^{-2} as seen in (9). This can be explained by the induced current paths (Fig. 7), which are mainly restricted to the wall layers for the horizontal field case, opposed to the vertical field case (Fig. 4). The flow in a diagonally oriented channel (with the gravity and magnetic field being aligned along the diagonal) shows something in between — see Fig. 8. This situation can be closer to the practically important circular cross-section.

Stability Analysis

In the case when the lateral walls are absent, i.e., $y_0 \to \infty$, the U(z) profile stability in the nonmagnetic situation was extensively studied previously [5], [6], [7] and many other relevant references are given in [3]. The stability of an electrically conducting fluid in the presence of a magnetic field was studied in [8] for the closely related situation of vertical flow between differentially heated vertical walls; the driving buoyancy force curl is identical to the present horizontal gap case, therefore the flow profile is the same. However, the situations are not fully identical because the temperature profiles are different.

The approach in the parallel flow linear stability analysis represents the perturbation fields for the velocity components, magnetic field, electrical potential, and temperature in the form

$$f=e^{\sigma t+i\alpha x}Y(y)Z(z)\,. \tag{15}$$

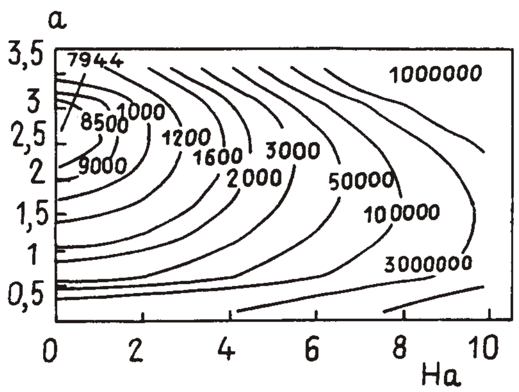

Fig. 9

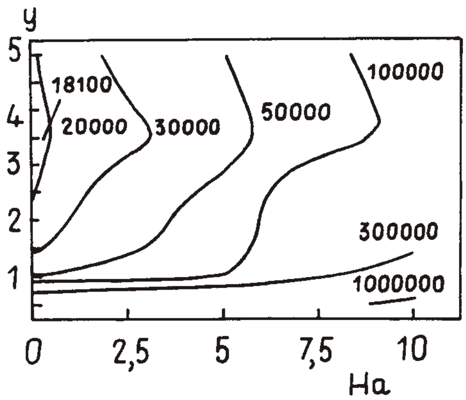

Fig. 10

Fig. 9. Critical Gr values for transversal rolls.

Fig. 10. Critical Gr values for longitudinal rolls.

Two cases are investigated: 1) the y-independent transversal rolls given by Y = 1, and the velocity v-component zero, 2) the x-independent longitudinal rolls when $a = 0$ in (15), all three velocity components are present, and it is possible to account for the finite size of the channel cross section.

1) In the case of a vertical magnetic field the z-component of the double curl of the momentum equations, accounting for the continuity equation, leads to the following equation:

$$\omega(w'' - a^2 w) = ia[-Uw'' + (U'' + a^2 U)w] + w'''' - (2a^2 + Ha^2)w'' + a^4 w \tag{16}$$

with the boundary conditions w = w' = 0 at z = ±1/2. Again transforming to the interval [−1, 1], w(z) is expanded as

$$w(z) = \sum_{n=1}^{N} w_n F_{10}(n,z)\,, \tag{17}$$

where the spectral functions satisfying the boundary conditions are

$$F_{10}(n,z) = \begin{cases} T_{k+3}(z) - (k+2)(k+4)/8\, T_3(z) + k(k+6)/8\, T_1(z)\,; & k = 2,4,\ldots \text{ even} \\ T_{k+3}(z) - (k+3)^2/4T_2(z) + (k+1)(k+5)/4\, T_0(z)\,; & k = 1,3,\ldots \text{ odd} \end{cases}. \tag{18}$$

Similarly to the stationary case discretization of Eq. (16) is made, yet for this problem the Gauss–Lobato (extrema of $T_{N-1}(z)$) grid helps to avoid spurious eigenvalues. The resulting algebraic generalized eigenvalue problem is solved by the ZGEGV routine from LAPACK.

The convergence of computed critical eigenvalues was found to be very good, e.g., in the reference nonmagnetic case N = 16, 32, 64, 132 results give the same critical Gr = 7944 for the wave number a = 2.6. The vertical magnetic field effect is to enhance the stability as shown in Fig. 9. The transition is for all computed parameters to the stationary state.

2) The x-independent rolls in a rectangular cross-section channel with the vertical magnetic field can be described by the following set of stability equations:

$$\begin{aligned} &\partial_t \Delta\psi = \Delta^2\psi - Ha^2\partial_{zz}\psi - Gr\partial_y\Theta \\ &\partial_t u + \partial_z\psi\partial_y U - \partial_y\psi\partial_z U = \Delta u + Ha^2(-\partial_y\varphi - u) \\ &\partial_t\Theta + u + \partial_z\psi\partial_y T_1 - \partial_y\psi\partial_z T_1 = Pe^{-1}\Delta\Theta \\ &-\partial_y u = \Delta\varphi\,, \end{aligned} \tag{19}$$

where $\partial_z\psi = v\,, \partial_y\psi = -w$.

The boundary conditions at the insulating walls are $\psi = \partial_n\psi = u = \partial_n\Theta = \partial_n\varphi = 0$, which are satisfied by expanding ψ in the spectral functions (18), u in (11), φ and Θ in (14). The discretization of the set (19) on the extrema grid leads to a real generalized eigenvalue problem which was solved by the routine DGEGV from the LAPACK. The results for N = 16 are presented in Fig. 10. The transition is supercritical leading to oscillating rolls for Pe = 0.01, however calculations for Pe = 0.1 showed transitions to stationary states for the square channel case. The main conclusions which can be derived from the present preliminary calculations are that the vicinity of walls i the square channel makes the flow more stable if compared to the wider channels, and the magnetic field is a greatly stabilizing factor.

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