# OPTIMIZATION OF AN MHD GENERATOR IN THE PRESENCE OF A CONSTRAINT ON THE DUCT DIVERGENCE ANGLE

A. I. El'kin

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The problem of minimizing the length of a MHD generator for a given power is investigated. In order to ensure the validity of the quasi-one-dimensional approximation a constraint is imposed on the divergence (convergence) angle of the duct. Mathematically, the problem is formulated as an optimal-response problem in the mathematical theory of optimal processes.

A critical Mach number, determined by the properties of the plasma, is introduced and it is established that the shape of the optimal duct depends significantly on whether the flow is sub- or super-critical, sub- or supersonic. It is shown that the modulus of the angle formed by the side walls of the duct with the duct axis always takes the maximum permissible value.

It follows from computer calculations that both expanding and contracting ducts are possible together with ducts with a single transition, i.e., ducts that first expand and then contract or first contract and then expand.

In using the quasi-one-dimensional approximation for analyzing MHD generators it is natural to confine oneself to the class of generators for which such a description is valid.

In particular, a limit should be imposed on the divergence angle of the duct [1]. This limit may also be associated with design considerations. In this case in optimizing MHD generators it is desirable to use mathematical methods of optimal control (nonclassical calculus of variations) [2]. As an example we will consider the problem of optimizing an MHD generator in a formulation similar to that of [3], i.e., we will require the generator to have the minimum length for a given power.

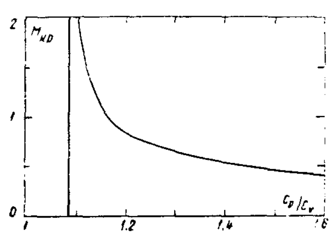

Fig. 1. Critical Mach number as a function of the adiabatic exponent for $\alpha$ = 12.

Here, as distinct from [3], it is assumed that the absolute value of the angle formed by the side walls of the duct does not exceed a certain limit.

We note that in the initial formulation the problem of [3] does not have a solution [4]. By imposing a constraint on the duct geometry in the form of a given length of the edge formed by adjacent walls, it is possible to solve the given problem by the methods of the classical calculus of variations [5].

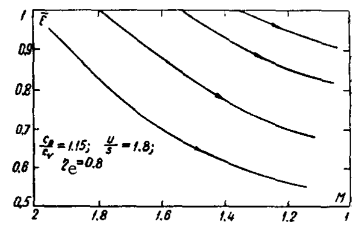

Fig. 2. Mach number as a function of the dimensionless duct width $\bar{b}$ for constant conductivity ($\alpha$ = 0) and supersonic regimes.

We write the system of equations describing in the quasi-one-dimensional approximation the plasma flow in the duct of an ideally segmented MHD generator of the Faraday type without friction and heat transfer:

$$\rho v \frac{dv}{dx} = -\frac{dp}{dx} + \sigma v B^2(\eta_e - 1), \quad (1)$$

$$\rho v \frac{d}{dx}\left(c_p T + \frac{v^2}{2}\right) = \sigma (vB)^2 \eta_e(\eta_e - 1), \quad (2)$$

$$p = \rho g R T, \quad (3)$$

$$\rho v b = G. \quad (4)$$

We take the temperature dependence of the conductivity in the form:

$$\sigma = \sigma_1 \left(\frac{T}{T_1}\right)^{\alpha}. \quad (5)$$

Moreover, we set $\eta_e$ = const.

We introduce the dimensionless quantities

$$\bar{v} = \frac{v}{v_1}, \quad \bar{b} = \frac{b}{b_1}, \quad \tau^* = \frac{T^*}{T^*_1}, \quad \tau = \frac{T}{T_1},$$

$$\bar{x} = \frac{x \sigma_1 B^2}{\rho_1 v_1}, \quad s = \frac{\sigma_1 B^2 b_1}{\rho_1 v_1},$$

where the subscript 1 corresponds to the parameters at the channel inlet, and the superscript * to the parameters of the adiabatically decelerated gas. After going over to dimensionless variables it is convenient to write Eqs. (1)–(5) in the form:

$$\frac{d\tau^*}{d\bar{x}} = \frac{2\tau^{\alpha}\eta_e(\eta_e - 1)\bar{b}\bar{v}^2}{1 + 2[M_1^2(k-1)]^{-1}} = \varphi_1, \quad (6)$$

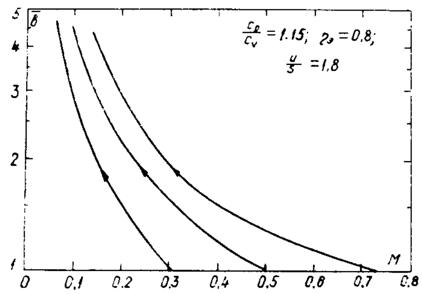


Fig. 3. Mach number as a function of the dimensionless duct width $\bar{b}$ for constant conductivity ($\alpha$ = 0) and subsonic regimes.

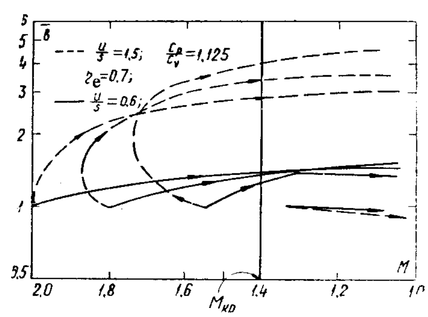


Fig. 4. Mach number as a function of the dimensionless duct width $\bar{b}$ for variable conductivity ($\alpha$ = 12) and supersonic regimes.

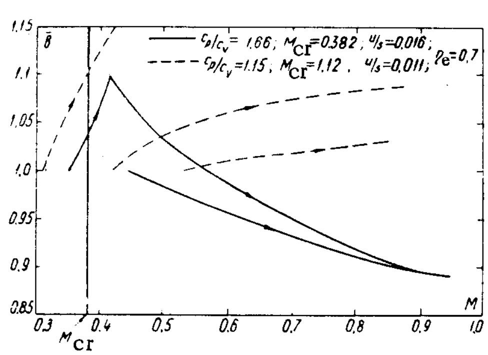


Fig. 5. Mach number as a function of the dimensionless duct width $\bar{b}$ for variable conductivity ($\alpha$ = 12) and subsonic regimes.

$$\frac{d\bar{v}}{d\bar{x}}=\frac{k\tau^{\alpha}\bar{b}\bar{v}(\eta_e-1)[1-\eta_e(k-1)/k]M_1^2+u\tau/s\bar{v}\bar{b}}{M_1^2-\tau\bar{v}^{-2}}=\varphi_2, \quad (7)$$

$$\frac{d\bar{b}}{d\bar{x}}=\frac{u}{s}=\varphi_3, \quad (8)$$

where $\tau=\tau^*[1+((k-1)/2)M_1^2]-[v^2M_1^2(k-1)/2]$. In these formulas G is the flow per unit height, b the duct width, u the tangent of the duct divergence angle. We will assume that the quantities $\tau^*$, $\bar{v}$, $\bar{b}$ are phase coordinates (it is convenient to treat them as projections of the three-dimensional vector Y), and that u/s is the control parameter. Then the problem is formulated as a problem of optimal response in the mathematical theory of optimal processes [2]. The control region U is given by the inequality

$$\left|\frac{u}{s}\right|\leqslant\left(\frac{u}{s}\right)_0. \quad (9)$$

It is assumed that the representative point is given at one end of the generator and the value of the phase coordinate $\tau^*$ at the other. Consequently, one end of the phase trajectory is fixed, while the other (moving) end must be located on the manifold N (the plane $\tau^*$ = const).

In accordance with [2], if $(y(\bar{x}), u/s(\bar{x}))$ is the optimal process, then there is a function $\psi(\bar{x})=(\psi_1(\bar{x}), \psi_2(\bar{x}), \psi_3(\bar{x}))$ such that the relations

$$\frac{d\psi_k(x)}{d\bar{x}}=-\frac{\partial H\left(\psi(\bar{x}), y(\bar{x}), \frac{u}{s}(\bar{x})\right)}{\partial y^k},$$

$$k=1,2,3; \quad (10)$$

$$\max H\left(\psi(\bar{x}), y(\bar{x}), \frac{u}{s}\right)=H\left(\psi(\bar{x}), y(\bar{x}), \frac{u}{s}(\bar{x})\right),$$

$$\frac{u}{s}\in U; \quad (11)$$

$$H\left(\psi(\bar{x}_1), y(\bar{x}_1), \frac{u}{s}(\bar{x}_1)\right)\geqslant 0 \quad (12)$$

are satisfied. The function H is given by

$$H\left(\psi, y, \frac{u}{s}\right)=\varphi_1\psi_1+\varphi_2\psi_2+\varphi_3\psi_3. \quad (13)$$

On substituting Eqs. (6)–(8) into Eq. (10), we obtain

$$\frac{d\psi_1}{d\bar{x}}=-\frac{\psi_2 u\bar{v}^3M_1^2[1+M_1^2(k-1)/2]}{s\bar{b}(\bar{v}^2M_1^2-\tau)^2}+$$

$$+\frac{\alpha\tau^{-1}+(1-\alpha)\bar{v}^{-2}M_1^{-2}}{(\bar{v}^2M_1^2-\tau)^2}\times$$

$$\times\psi_2 k\bar{b}(1-\eta_e)[1-\eta_e(k-1)/k]\times$$

$$\times(1+M_1^2(k-1)/2)\tau^{\alpha}M_1^4\bar{v}^5+$$

$$+\psi_1\eta_e(1-\eta_e)\bar{b}\bar{v}^2\alpha\tau^{\alpha-1}(k-1)M_1^2, \quad (14)$$

$$\frac{d\psi_2}{dx}=\psi_2\frac{u[\bar{v}^4M_1^4(k-1)+\tau M_1^2\bar{v}^2+\tau^2]}{s\bar{b}(\bar{v}^2M_1^2-\tau)^2}+$$

$$+\frac{\psi_2 k\bar{b}(1-\eta_e)[1-\eta_e(k-1)/k]\bar{v}^4M_1^4}{(\bar{v}^2M_1^2-\tau)^2}\times$$

$$\times\left\{\alpha\tau^{\alpha-1}(1-k)(\bar{v}^2M_1^2-\tau)-\right.$$

$$\left.-\frac{\tau^{\alpha}}{M_1^2}\left[\frac{3\tau}{v^2}+M_1^2(k-2)\right]\right\}+$$

$$+\psi_1\frac{2\eta_e(\eta_e-1)\bar{b}}{1+2[M_1^2(k-1)]^{-1}}[\alpha\tau^{\alpha-1}M_1^2(k-1)\bar{v}^3-2v\tau^{\alpha}], \quad (15)$$

$$-\frac{d\psi_3}{d\bar{x}}=\psi_2\frac{u\tau\bar{v}}{s\bar{b}^2[\tau-\bar{v}^2M_1^2]}+$$

$$+\frac{\psi_2 k\tau^{\alpha}\bar{v}^3M_1^2(\eta_e-1)[1-\eta_e(k-1)/k]}{\bar{v}^2M_1^2-\tau}+$$

$$+\frac{\psi_1 2\tau^{\alpha}\eta_e(\eta_e-1)\bar{v}^2}{1+2[M_1^2(k-1)]^{-1}}. \quad (16)$$

In the case considered H is a linear function of u/s, the coefficient F of u/s being equal to

$$F=\psi_3+\psi_2\tau\bar{v}^{-1}\bar{b}^{-1}(M_1^2-\tau\bar{v}^{-2})^{-1}. \quad (17)$$

Consequently, optimality condition (11) must take the form:

$$\frac{u}{s}=\left(\frac{u}{s}\right)_0 \operatorname{sign} F. \quad (17a)$$

We will write the transversality condition for the moving end of the phase trajectory. It consists in the orthogonality of the vector $\psi(\bar{x}_1)$ to the tangent vectors of the manifold $N_1$ (the plane $\tau^*_1$ = const) at the point $\bar{x}_1$ [2]. The required condition has the form:

$$\psi_3(\bar{x}_1)=\psi_2(\bar{x}_1)=0. \quad (18)$$

From Eqs. (12), (13), and (18), there follows $\psi_1(\bar{x}_1)\varphi_1(\bar{x}_1)\geq 0$.

Here, $\varphi_1(\bar{x}_1)<0$, and $\psi_1(\bar{x}_1)\neq 0$, since the functions $\psi$ do not vanish simultaneously. Consequently, $\psi_1(\bar{x}_1)<$ $<0$.

The functions $\psi$ have been determined correct to a common constant positive multiplier. Therefore we can set

$$\psi_1(\bar{x}_1)=-1. \quad (19)$$

Equations (6)–(8), (14)–(16), (17a) with initial conditions $\psi_3(\bar{x}_1)=\psi_2(\bar{x}_1)=0$; $\psi_1(\bar{x}_1)=-1$; $\bar{b}_1=\bar{v}_1=\tau^*_1=1$ make it possible to calculate the optimal trajectories. However, at the point $\bar{x}_1 F=0$ and the sign of the control cannot be determined from Eq. (17a). We will find the control at a point infinitely close to $\bar{x}_1$. Obviously, in this case $u/s=(u/s)_0$ sign dF.

Differentiating Eq. (17), by means of Eqs. (15), (16) and the initial conditions we obtain

$$\frac{u}{s}=\left(\frac{u}{s}\right)_0 \operatorname{sign}\left\{\frac{1+M^2[1-\alpha(k-1)]}{1-M^2}d\bar{x}\right\}. \quad (20)$$

Here, $d\bar{x} > 0$ for fixed conditions at the generator outlet (the left end of the phase trajectory is the moving end) and $d\bar{x} < 0$ for given conditions at the inlet. The equation obtained is used to determine the control at the point $\bar{x}_1$ and can be interpreted as the optimality condition of the "elementary" MHD generator. It follows from (20) that the control changes sign at M = 1 and the "critical" Mach number

$$M_{cr} = [\alpha(k-1) - 1]^{-1/2}. \tag{21}$$

Relation (21) is shown in Fig. 1 for $\alpha = 12$.

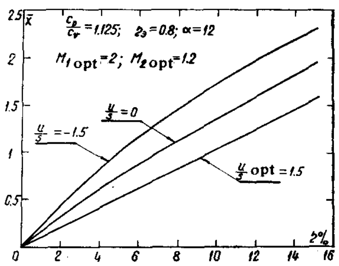


Fig. 6. Comparison of optimal MHD generator with generators of other types.

Equation (20) can also be obtained without introducing the maximum principle (11). It follows from Eq. (6) that the quantity $-\left.\frac{d\tau}{d\bar{x}}\right|_{\bar{x}=\bar{x}_1}$ (the specific linear power) does not depend on u/s. However, the quantity $-(d^2\tau^*/d\bar{x}^2)\,d\bar{x}$ (the specific linear power increment over the length of the elementary generator) does depend on u/s and in the optimal generator should be a maximum. (Here $d\bar{x} > 0$, if the conditions at the inlet are given, and $d\bar{x} < 0$ for given conditions at the outlet.) Differentiating Eq. (6), using Eqs. (7), (8) and substituting $\bar{b} = \bar{v} = \tau = 1$, we see that $-(d^2\tau^*/d\bar{x}^2)\,d\bar{x}$ depends linearly on u/s, and correct to the sign the coefficient of u/s is equal to the expression in braces in Eq. (20). Consequently, the maximum specific linear power increment is obtained for condition (20).

The optimal phase trajectories of an MHD generator of finite length were calculated numerically on a "Ural-4" computer. The calculations were made for different values of the parameters u/s; $\alpha$(0, 12); $\eta_e$ (0.7–0.85); $M_1$ (2–0.3); $c_p/c_v$ (1.66–1.11).

Altogether we calculated about 500 variants. For constant conductivity ($\alpha = 0$) and with given conditions at the outlets (moving end on the left) the channel contracts toward the outlet in supersonic regimes (Fig. 2) and expands in subsonic regimes (Fig. 3).

For variable conductivity ($\alpha = 12$), given conditions at the outlet and the supersonic regime (Fig. 4) the ducts are expanding for $M > M_{cr}$ and contracting for $M < M_{cr}$. However, if $M_1 > M_{cr}$ at the channel inlet, while at the outlets $M_2 < M_{cr}$, both expanding ducts and ducts with a transition, i. e., first expanding and then contracting, are possible. We note that the transition always occurs at $M < M_{cr}$. At the transition point the quantity $d\bar{b}/d\bar{x}$ experiences a discontinuity, but in practice the transition should probably be made smooth. It is possible to take $u^1 = d^2\bar{b}/d\bar{x}^2$ as the control with the constraint $|u^1| \le u_0^1$, and treat $u/s = d\bar{b}/d\bar{x}$ as the phase coordinate, on which a constraint is also imposed. In this case the duct profile will be smooth, but the problem is considerably complicated. We also note that if at the outlet the parameters are given so that $M_2 > M_{cr}$, the duct is always expanding.

For variable conductivity ($\alpha = 12$), given conditions at the outlet, and subsonic regimes (Fig. 5) the ducts expand toward the outlet when a combustion-product plasma is used ($c_p/c_v = 1.11$–$1.15$). In this case (Fig. 1) $M < M_{cr}$. For inert gases ($c_p/c_v = 1.66$) the ducts are expanding at $M < M_{cr}$ and contracting at $M > M_{cr}$. If $M_1 < M_{cr}$ a transition is possible at $M > M_{cr}$.

In conclusion we note that a comparison of the optimal MHD generator with generators of other types (constant cross section, contracting or expanding with straight generatrices) confirm the optimality of the calculated MHD generator in the above-mentioned sense. Thus, from Fig. 6, which offers an example of such a comparison, it is clear that for a sufficiently large conversion factor $\eta = 1 - \tau^* = 14\%$ and at the same value of $|u/s|_0$ the optimal MHD generator (expanding) is shorter than the generator with a contracting duct by a factor of 1.5 and shorter than the generator with a constant cross section by a factor 1.25.

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