MAGNETIC FIELD EXCITATION BY A SYSTEM OF SUBMERGED JETS

A. Gailitis

Magnitnaya Gidrodinamika, Vol. 5, No. 3, pp. 31–37, 1969

UDC 538.4

It is shown that the motion of three or more nonintersecting jets in an electrically homogeneous fluid conductor may create a magnetic field. The magnetic Reynolds numbers necessary for self-excitation are calculated.

According to contemporary ideas, the magnetic fields of the earth, sun, and a number of other heavenly bodies are created by the mechanical motion of electrically conducting material. The conditions under which the hydrodynamic motion of an electrically homogeneous conductor excites a magnetic field have been discussed in this connection. It turns out that the geometrically simplest motions, among which are axisymmetric motions [1] and two-dimensional motions, do not always ensure self-excitation. The conditions necessary for self-excitation are satisfied by such special motions as the convection patterns of Ballard [2] and Braginskii [3, 4] as well as the rather complicated flows which may be treated as turbulence with certain spatially averaged properties [5–7]. It is thus of interest to find out which of the simplest motions could ensure self-excitation. One of these motions was considered by Herzenberg [8, 9]. Another possible pattern is discussed below.

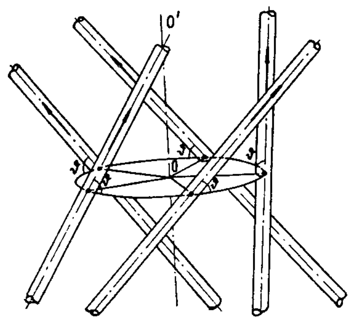

Fig. 1. Position of the jets.

Preliminary estimates. We consider an unbounded space, filled with a motionless fluid of electrical conductivity σ, in which n jets of the same fluid are submerged. A jet is understood to be a cylindrical region of radius R moving as a whole along its own axis with a velocity v. The system of jets (Fig. 1) is symmetric when all the jets are situated at equal distances λ from the axis of symmetry OO' and when the configuration after rotation of the system about the symmetry axis through an angle $2\pi/n$ is the same as the initial configuration. All the jets subtend the same angle $\vartheta \neq 0$ with the plane which is normal to the axis of symmetry; thus they do not intersect, and each passes at a distance $L = 2\lambda \sin\vartheta/(\mathrm{ctg}^2(\pi/n) + \sin^2\vartheta)^{1/2}$ from its neighbors. To simplify the calculations, the jets are assumed to be thin: $R \ll L$.

We estimate the order of magnitude of the velocities for which self-excitation of the magnetic field is possible in this system. For this purpose we consider the particular case n = 3 and $\vartheta = \arcsin(1/\sqrt{3}) \approx 35.265°$, when all the jets are mutually perpendicular. If there is a uniform magnetic field B_0 normal to one of the jets, an induced current j (Fig. 2) will flow in this jet and in the surrounding medium. This current will create its own field B_1, parallel to the jet and equal to

$$B_1 = B_0\sigma\mu_0 v R^2/(2L) \qquad (1)$$

at the closest point of the neighboring jet.

If B_1 were equal to (1) at all points of the neighboring jet and the requirement

$$\sigma\mu_0 v R^2/L > 2, \qquad (2)$$

were satisfied, the self-excitation would be such that each jet would rotate the field by 90°, amplifying it at the same time, and after three turns the field would return to its initial direction.

However, the field B_1 is strongly inhomogeneous, and the three-stage picture described above is only qualitative, condition (2) being clearly an understatement. Systematic calculations are required to make it more exact.

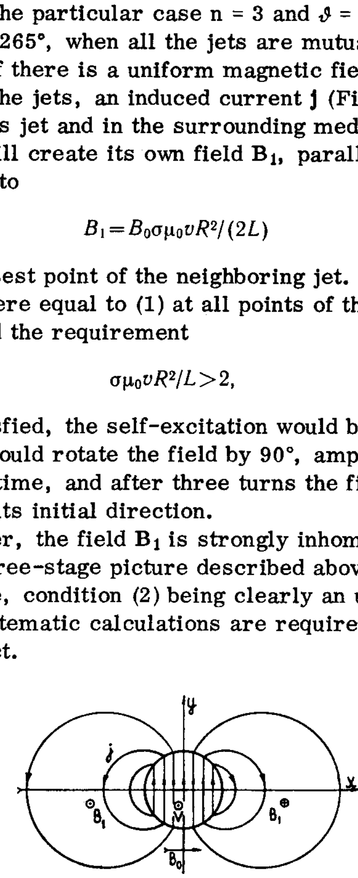

Fig. 2. Direction of the induced current and the magnetic field it creates.

Derivation of an integral equation. Condition (2) may be made more exact by solving the equation

$$\mathrm{rot}\,\mathbf{B} = \mu_0\mathbf{j} = \mu_0\sigma([\mathbf{v}\times\mathbf{B}] - \mathrm{grad}\,\Phi), \qquad (3)$$

which, after eliminating the electrostatic potential,

$$\triangle\mathbf{B} = -\mu_0\sigma\,\mathrm{rot}\,[\mathbf{v}\times\mathbf{B}] \qquad (4)$$

can be written in the integral form

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0\sigma}{4\pi}\mathrm{rot}\int\frac{d^3r'}{|\mathbf{r}-\mathbf{r}'|}[\mathbf{v}\times\mathbf{B}(\mathbf{r}')] = -\frac{\mu_0\sigma}{4\pi}\int\frac{d^3r'}{|\mathbf{r}-\mathbf{r}'|^3}\times$$
$$\times\{\mathbf{v}[\mathbf{B}(\mathbf{r}')\mathbf{r} - \mathbf{B}(\mathbf{r}')\mathbf{r}'] - \mathbf{B}(\mathbf{r}')(\mathbf{v}\mathbf{r} - \mathbf{v}\mathbf{r}')\}. \qquad (5)$$

Expression (5) is in a suitable form since the integration here extends only through the volume of the jets (outside them $\mathbf{v} = 0$) and the field $\mathbf{B}(\mathbf{r})$ may be represented as the sum of the fields created by each jet separately. The field inside one of the jets can be determined in two stages, by calculating the field produced by the other jets (the external field $\mathbf{B}_a$), and calculating the self-field $\mathbf{B}_i$ of a given jet.

Since the jet is thin ($R \ll L$), $\mathbf{r}$ and $\mathbf{r}'$ can be replaced by the coordinates of the jet axes in determining the external field in (5), and the integration can be carried out over the stream cross section ($d^2\rho'$) leaving the integration with respect to its length $dl'(d^3\mathbf{r}' = d^2\rho' dl')$. Denoting the average field over the cross section by

$$\mathbf{b}(l') = \frac{1}{\pi R^2} \int d^2\rho' \mathbf{B}(l', \rho'), \tag{6}$$

we can write (5) in the form

$$\mathbf{B}_a(\mathbf{r}) = -\frac{\mu_0 \sigma R^2}{4} \sum{}' \int \frac{dl'}{|\mathbf{r}-\mathbf{r}'|^3} \times \\ \times \{\mathbf{v}[\mathbf{b}(l')\mathbf{r} - \mathbf{b}(l')\mathbf{r}'] - \mathbf{b}(l')(\mathbf{vr} - \mathbf{vr}')\}. \tag{7}$$

Here Σ' denotes summation over all jets except the one under consideration.

In what follows we determine only the conditions for self-excitation of a symmetric field which coincides with itself on being rotated through an angle $2\pi/n$ about the symmetry axis. This field assumes identical values at symmetric points of the various jets, and the problem is reduced to determining $\mathbf{B}_a(l)$ and $\mathbf{b}(l)$ for some single jet. Expression (7) assumes the form

$$B_{a\alpha}(l) = \frac{R R_m}{L} \int \frac{dl'}{L} \sum_{\beta=1}^{2} K_{\alpha\beta}\left(\frac{l}{L}, \frac{l'}{L}\right) b_\beta(l'), \tag{8}$$

where

$$R_m = \mu_0 \sigma v R, \tag{9}$$

and the kernel of Eq. (8), i.e., $K_{\alpha\beta}(l, l')$, is a cumbersome expression which results from passing to coordinates associated with the selected jet in (7). It is a 2×2 matrix since it connects the two components of the vector $\mathbf{B}_a$ normal to the jet with the two normal components of $\mathbf{b}$.

Calculating the self-field of the jet. Equation (8) can be used to solve the first half of the problem, i.e., to determine the field acting on one of the jets and produced by the remaining jets. To reduce (8) to a closed form, the field averaged over the jet cross section $\mathbf{b}(l')$ must be expressed in terms of the external field $\mathbf{B}_a(l)$ acting on the jet.

Since the jet is thin ($L \gg R$) we can make the approximation that $\Delta \mathbf{B}_a = 0$ and that $\mathbf{B}_a$ is exclusively a function of z. Inserting $\mathbf{B} = \mathbf{B}_a + \mathbf{B}_i$ into (4), we have

$$\Delta \mathbf{B}_i = -\mu_0 \sigma \operatorname{rot}[\mathbf{v} \times (\mathbf{B}_a + \mathbf{B}_i)]. \tag{10}$$

After introducing the vector potential from $\mathbf{B}_i = \operatorname{rot} \mathbf{A}$, Eq. (10) transforms to

$$(\Delta + \mu_0 \sigma \mathbf{v} \times \operatorname{rot}) \mathbf{A} = -\mu_0 \sigma [\mathbf{v} \times \mathbf{B}_a]. \tag{11}$$

In the coordinates of Fig. 2, where z is in the direction of the jet, expression (11) means that

$$\Delta A_z = 0,$$

which gives $A_z = 0$ and

$$(\Delta - \mu_0 \sigma v \partial/\partial z) A_x = \mu_0 \sigma v B_{ay}(z), \\ (\Delta - \mu_0 \sigma v \partial/\partial z) A_y = -\mu_0 \sigma v B_{ax}(z). \tag{12}$$

Since

$$B_{ix} = -\partial A_x/\partial z, \quad B_{iy} = \partial A_y/\partial z, \tag{13}$$

B_{ix} is expressed in terms of B_{ax} only, and B_{iy} in terms of B_{ay} and it is sufficient to consider only one of the equations in (12) (the first), omitting the x and y subscripts.

The solution of (12) can be expressed by means of the Green's function

$$A(z, r) = \int_{-\infty}^{\infty} G(z - z', r) B_a(z') dz'. \tag{14}$$

The quantity b(z) is needed for substitution into (8). It is associated with A by

$$b(z) = B_a(z) + \frac{2}{R_0^2} \frac{\partial}{\partial z} \int_0^R A(z, r) r dr \tag{15}$$

or

$$\mathbf{b}(z) = \int_{-\infty}^{\infty} \gamma(z') \mathbf{B}_a(z - z') dz', \tag{16}$$

where

$$\gamma(z) = \delta(z) + \frac{2}{R_0^2} \frac{\partial}{\partial z} \int_0^R G(z, r) r dr. \tag{17}$$

G(z, r) satisfies the equation

$$\left(\Delta - \mu_0 \sigma v \frac{\partial}{\partial z}\right) G(z, r) = \mu_0 \sigma v \delta(z). \tag{18}$$

When $R \ll L$, it follows from (2) that $\mu_0 \sigma v R \gg 1$ and (18) can be simplified:

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \mu_0 \sigma v \frac{\partial}{\partial z}\right) G(z, r) = \mu_0 \sigma v \delta(z), \quad r < R, \\ \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) G(z, r) = 0, \quad r > R. \tag{19}$$

To solve (19) it is convenient to use the Fourier transform:

$$G(z, r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk G_k(r) e^{ikz}, \\ \gamma(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \gamma_k e^{ikz}, \quad \delta(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikz} dk, \tag{20}$$

which transforms (19) to

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-ik\mu_0\sigma v\right)G_k(r)=\mu_0\sigma v,\qquad r<R,$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-k^2\right)G_k(r)=0,\qquad r>R. \tag{21}$$

A solution of (21) which decreases for large values of the argument is

$$G_k(r)=\frac{i}{k}\left[1+C_iJ_0(\sqrt{y}r/R)\right],$$
$$r\leqslant R,\quad y=-ik\mu_0\sigma vR^2, \tag{22}$$

$$G_k(r)=\frac{i}{k}C_aK_0(|kr|),\qquad r\geqslant R. \tag{23}$$

The boundary conditions

$$G_k(R+0)=G_k(R-0),$$
$$\partial/\partial rG_k(r)\,|_{r=R+0}=\partial/\partial rG_k(r)\,|_{r=R-0} \tag{24}$$

are used to determine the constants of integration

$$C_i=-\left\{J_0(\sqrt{y})+\sqrt{y}\left[\frac{1}{2}\ln\frac{(kR)^2}{4}+C\right]J_1(\sqrt{y})\right\}^{-1},$$

$$C_a=C_i\sqrt{y}J_1(\sqrt{y}). \tag{25}$$

Here C = 0.5772 is Euler's constant.

In deriving (25) we assumed that the jet is thin, which means that $kR \ll 1$ and

$$K_0(|kr|)\approx -C-\frac{1}{2}\ln\frac{(kr)^2}{4}. \tag{26}$$

In accordance with (21), (24), (26), and (29)

$$\gamma_k=1+\frac{2ik}{R^2}\int_0^R r\,dr\,G_k(r)=-\frac{2C_i}{R^2}\int_0^R r\,dr\,J_0(\sqrt{y}\,r/R)=$$
$$=2\sqrt{y}\,J_0(\sqrt{y})/J_1(\sqrt{y})+2y\left[C+\frac{1}{2}\ln\frac{(kR)^2}{4}\right], \tag{27}$$

$$\gamma(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{ikz}dk\left(\frac{\sqrt{y}\,J_0(\sqrt{y})}{J_1(\sqrt{y})}+\right.$$
$$\left.+y\left[C+\frac{1}{2}\ln\frac{(kR)^2}{4}\right]\right)^{-1}. \tag{28}$$

Thus the calculation of the self-field of a jet reduces to Eq. (16) with the kernel given in (28). Expressions (16) and (28) together with Eq. (8) form a closed system for determining (9).

The use of (28) is complicated by the fact that the integrand contains an oscillatory factor exp ikz. The situation can be improved by shifting the path of integration into the complex k-plane (Fig. 3).

For $z > 0$, path (1) must be shifted into the upper half-plane (2), and for $z < 0$, to the lower half-plane (3). It is significant that the integrand contains a factor $\ln(kR)^2$ and so has a branch point at $k = 0$, to which two branch cuts must be taken from $\pm i\infty$. If the branch cuts coincide with the positive and negative imaginary axes, the integral is reduced to integration along the boundaries of these branch cuts and gives different expressions for positive and negative z:

$$\gamma(z)=\frac{1}{RR_m}\int_0^{\infty}dy\,e^{-\frac{yz}{RR_m}}\times$$
$$\times\left\{\left[\frac{J_0(\sqrt{y})}{J_1(\sqrt{y})}+\sqrt{y}\left(\ln\frac{y}{2R_m}+C\right)\right]^2+\frac{\pi^2}{4}y\right\}^{-1},\quad z>0, \tag{29}$$

$$\gamma(z)=-\frac{1}{RR_m}\int_0^{\infty}dy\,e^{-\frac{yz}{RR_m}}\times$$
$$\times\left\{\left[\frac{I_0(\sqrt{y})}{I_1(\sqrt{y})}-\sqrt{y}\left(\ln\frac{y}{2R_m}+C\right)\right]^2+\frac{\pi^2}{4}y\right\}^{-1},\quad z<0. \tag{30}$$

Calculation of the eigenvalue. The conditions for self-excitation were determined by numerically solving system (8) and (16) with the kernel (29) and (30) on a computer.

The kernel of (29) and (30) was integrated by means of Simpson's formula with a varying step length (304 steps). The infinite integration interval in (8) and (16) was divided into two parts, i.e., $|l'| < aL$ and $|l'| > aL$. In the inner region (16 steps) the integration was carried out with respect to the variable l', and in the outer region (16 steps) with respect to $1/l'$. In both regions Simpson's formula with a constant step length was employed, reducing the integral equations (8) and (16) to matrix equations. A change in scale a showed that the selected number of steps led to an accuracy of no less than 1%. The results obtained are shown in Fig. 4, where the critical magnetic Reynolds number R_m is given as a function of the ratio L/R for several ϑ and n. The reverse order was used in making the calculations, i.e., the values of R_m, ϑ, and n were set and L/R subsequently calculated. Since the problem is nonlinear in the ratio L/R (exponents in (29) and (30) contain z/R, while the scale of integration with respect to z is equal to aL), this ratio was determined by an iteration method: the initial value of L/R was used in (29) and (30) to calculate the eigenvalue of L/R for Eq. (8),* after which the value of L/R from (29) and (30) was made more exact.

Figure 4 shows that for values of R_m sufficiently large, self-excitation of the magnetic field is possible in our system of submerged jets. For each n number of jets there exists an optimum angle (depending on L/R) for which R_m is a minimum (curves for ϑ close to the optimum are given in Fig. 4).

Self-excitation is possible for all the numbers of jets investigated ($n \geq 3$), but as n decreases the critical value of R_m rapidly increases. It should also be stressed that the treatment employed is legitimate for $L/R \gg 1$,

*The eigenvalue of the matrix equation (8) and (16) was calculated in turn by means of an internal iteration cycle by correcting a table of values of b and B.

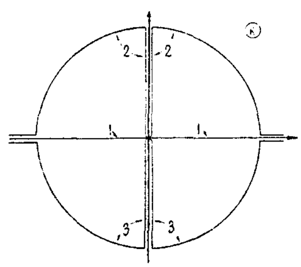

Fig. 3. Displacement of the integration path: 1) initial path; 2) path for $z > 0$; 3) path for $z < 0$.

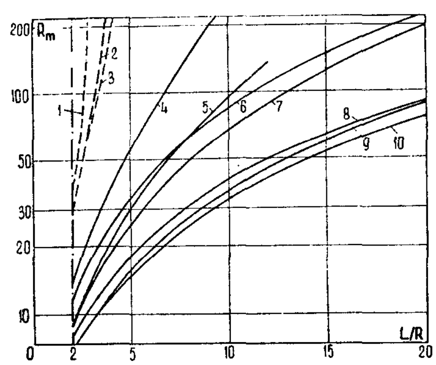

Fig. 4. Critical values of R_m as a function of L/R: 1) 3 jets with $\vartheta = 25°$; 2) 3 jets with $\vartheta = 20°$; 3) 3 jets with $\vartheta = 15°$; 4) 4 jets with $\vartheta = 20°$; 5) 5 jets with $\vartheta = 35°$; 6) 5 jets with $\vartheta = 15°$; 7) 5 jets with $\vartheta = 25°$; 8) 7 jets with $\vartheta = 20°$; 9) 7 jets with $\vartheta = 40°$; 10) 7 jets with $\vartheta = 30°$.

but for values of L/R which are not large the curves in Fig. 4 have a high systematic error. For n = 3 all the calculated curves lie in this region, and so they are drawn in dashed lines. As regards the fundamental question of whether self-excitation of a magnetic field is possible with three mutually perpendicular jets ($n = 3, \vartheta \approx 35.265°$), no solution could be obtained since the calculations led to such high values of R_m that the present numerical method turned out to be inapplicable.

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