SELF-EXCITATION OF A MAGNETIC FIELD BY A PAIR OF ANNULAR VORTICES

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Two steady-state annular vortices having a common axis of symmetry are considered in an unbounded electrically conducting fluid. An analytical solution is given for the self-excitation of a magnetic field by such a system for the limiting case in which the vortices are thin. The conditions for self-excitation are found. If the vortices are situated far from each other, a dipole moment is excited when the fluid rotates in the same direction; a quadripole moment is excited when the fluid rotates in opposite directions. As the vortices approach each other the multipolarity field increases.

It has been shown by Tverskoi [1] that a ring vortex in an electrically conducting fluid can excite a magnetic field **B**. Although the motion in the vortex is axisymmetric, the excited field is asymmetric. Thus the result does not contradict Cowling's [2] theorem on the impossibility of exciting a magnetic field axisymmetric ally. The possibility of excitation was shown in [1] with a vortex excited periodically in the presence of an external boundary.

Here we examine a steady-state vortex pair in an unbounded volume of uniform incompressible fluid of electrical conductivity σ.

The vortex pair consists of two similar vortex rings I and II having a common symmetry axis (Fig. 1). The fluid in each ring moves in meridian planes only; i. e., in planes passing through the axis of symmetry. The motion is the same in all these planes and occurs in concentric paths r = const.* The only nonzero component of velocity is v_χ, which is a function of both r and χ. Because of the incompressibility of the fluid the product $\rho v_\chi(r, \chi)$ is a function of r only. To simplify the calculations it is assumed that the rings are thin, i. e., $v_\chi \neq 0$ only when $r < R$, and that the small radius of the ring R is much less than both the axial radius a and the distance between the rings z_0.

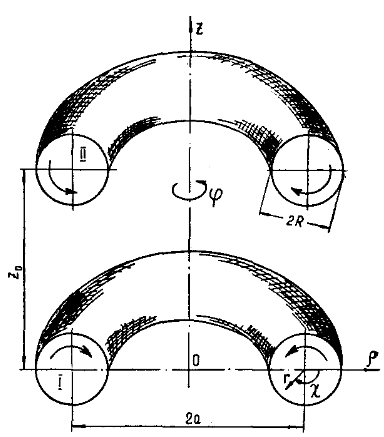

Fig. 1. Diagram of the vortex configuration.

Self-excitation occurs when the equation

$$\mathrm{rot}\,\mathbf{B} = \mu_0\sigma([\mathbf{v}\times\mathbf{B}] - \mathrm{grad}\,\Phi) \tag{1}$$

*Two systems of orthogonal coordinates are used: inside each ring a toroidal system r, χ, φ is used while for the full space the cylindrical system ρ, φ, z is used. φ is the azimuthal angle in the meridian plane, and r, χ are the polar coordinates in this plane with center r = 0 on the central line of the given ring; $\rho = r\cos\chi + a$ and $z = -r\sin\chi$ in the first ring and $z = z_0 - r\sin\chi$ in the second.

has a nontrivial solution $B \neq 0$ for the boundary condition $B \to 0$. The electrostatic potential Φ is eliminated from (1) by employing the curl operation:

$$\Delta \mathbf{B} = -\mu_0 \sigma \, \mathrm{rot}\, [\mathbf{v} \times \mathbf{B}]. \tag{2}$$

Equation (2) can be easily represented in integral form:

$$\mathbf{B} = \frac{\mu_0 \sigma}{4\pi} \int \frac{d^3 r'}{|\mathbf{r} - \mathbf{r}'|} \mathrm{rot}\, [\mathbf{v}(\mathbf{r}') \times \mathbf{B}(\mathbf{r}')]. \tag{3}$$

The integration in (3) is made over the volume of the rings only, since $v = 0$ outside. Thus only the field inside the rings need be calculated, and it can be represented as a sum of the self-field of the ring in question and the field of the other ring.

Because of axial symmetry the eigen solutions of (3) can depend on φ only through the factor $e^{im\varphi}$, created by ring II on the central line $r = 0$ of ring I, can be represented by $B_0 e^{im\varphi}$.

When $r \neq 0$ the self-field of ring I must be added to this field. The nature of the total field can be estimated if we take into account result (10) given below: for thin rings ($R \ll a$) self-excitation occurs only when rotation is rapid or, more exactly, for a magnetic Reynolds number $Rm \gg 1$. When Rm is large we can use the concept of "freezing" of the magnetic field lines in the fluid, according to which a rapid rotation inside ring I leads to the following:

(a) B_r is averaged over the angle χ and, since div $B = 0$,

$$B_r = -irmB_0 e^{im\varphi}/(2a); \tag{4}$$

(b) a nonnegligible component B_χ is formed; however, it is unimportant for the calculation since it does not appear in the product $[\mathbf{v} \times \mathbf{B}]$;

(c) the distribution of the field B_φ is established over the cross section of the ring:

$$B_\varphi = \rho B_0 e^{im\varphi}/a. \tag{5}$$

Formula (5) is an asymptotic expression for $Rm \gg 1$ and $R \ll a$, z_0. It can be obtained by considering the rotation of the azimuthal distance between two elements $dl_\varphi = \rho d\varphi$, which varies in proportion to ρ, and by using the principle of "freezing in" $B_\varphi \sim dl_\varphi$ [3].

Inserting (4), (5) into (3) we can determine the field $\mathbf{B}^1$, generated by ring I, at each point ρ, φ, z and particularly on the central line $\rho = a$, $z = z_0$ of ring II. To do this $g \equiv \mathrm{rot}[\mathbf{v}(\mathbf{r}') \times \mathbf{B}(\mathbf{r}')]$ must be written in toroidal coordinates:

$$g_{\chi'} = \frac{im}{\rho'} v_{\chi'} B_{\varphi'} + \frac{1}{\rho'} \frac{\partial}{\partial r'} \rho' v_{\chi'} B_{r'}, \qquad g_{r'} = g_{\varphi'} = 0 \tag{6}$$

and the result must be expressed in unit vectors at the observation point:

$$g_\varphi = g_{\chi'} \sin \chi' \cdot \sin (\varphi - \varphi') \tag{7}$$

(g_ρ and g_z are unimportant in what follows).

When (3) is integrated with respect to χ' (allowing for the fact that the ring is thin) the azimuthal component of the field on the central line of the second ring is obtained:

$$B_\varphi(a, \varphi, z_0) = -\mu_0 \sigma a^{-2} B_0 e^{im\varphi} \int_0^R v_\chi r^2 dr / T_m(z_0/a), \tag{8}$$

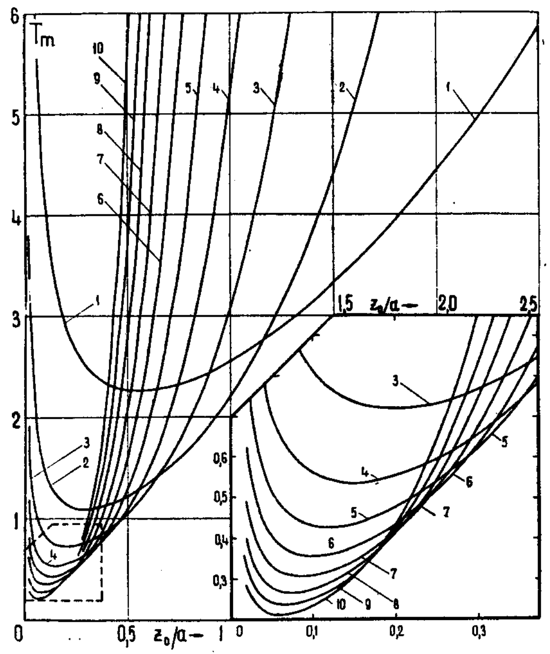

Fig. 2. T_m as a function of z_0/a. The numbers by the curves indicate the values of m. The portion surrounded by a dashed line is shown in fourfold magnification.

$$T_m^{-1}(z_0/a) = \frac{ma^2}{2} z_0 \oint d\varphi' \frac{\sin(\varphi'-\varphi)\sin m(\varphi'-\varphi)}{|r-r'|^3_{\rho=\rho'=a,\, z-z'=z_0}}. \tag{9}$$

If z_0 is replaced by $-z_0$ and $B_0 e^{im\varphi}$ by $B_\varphi(a, \varphi, z_0)$ and if the sign of v_χ made to agree with the sense of rotation in the second ring, then (8) and (9) define the field of ring II on the central line of ring I. It will be equal to $B_0 e^{im\varphi}$ (with self-excitation of the fields ensured) when the following two conditions are fulfilled;

(a) The rotation of the rings is in the opposite sense;

(b)

$$\mathrm{Rm} \equiv \mu_0 \sigma \int_0^R (r/R)^2 v_\chi dr = \pm (a/R)^2 T_m(z_0/a). \tag{10}$$

Thus self-excitation of the magnetic field arises when Rm has the value of (10). Since this is first attained with the smallest value of T_m, a magnetic field is excited with the corresponding value of m (dominant for a given z_0/a).

T_m is expressed in terms of the complete elliptic integrals E and K, in particular,

$$T_1 = kaz_0^{-1}[(2-k^2)K(k) - 2E(k)]^{-1}, \tag{11}$$

where $k = (1 + (z_0/2a)^2)^{-1/2}$. However, there is a different expression for each m, and as m increases the formulas become more complicated. The values of T_m in Fig. 2 were calculated by another method, with which T_m is expressed in terms of the hypergeometric function

$$T_m = \frac{4(m-1)!\Gamma(3/2)(k/2)^{1-2m}}{\pi mk(1-k^2)^{1/2}\Gamma(m+1/2)} \Big/ F(m+1/2, m+1/2; 2m+1; k^2) \tag{12}$$

and F is determined by summing the series on an electronic computer.

It is characteristic of the curves in Fig. 2 to mutually intersect each other. In other words, within the various regions of change in z_0/a, we have various predominant values: for $z_0 > 1.16a$, m = 1; for $1.16\,a > z_0 > 0.65a$, m = 2; for $0.65a > z_0 > 0.45a$, m = 3; etc. As $z_0/a \to 0$, the dominant value of m increases as $m \sim a/z_0$.

The two signs ± in (10) correspond to the two possible directions of rotation. Although the resulting patterns of the magnetic field and electric currents are different, self-excitation is obtained in both cases for the same absolute value of Rm. When the sign is negative, i. e., for the directions of rotation shown in Fig. 1, the field B_φ for fixed values of φ has the same direction in both rings, while for rotation with the positive sign, the directions of B_φ are mutually opposed.

In the first case part of the current lines link both rings and in the second lines of electric current form two closed systems around each of the vortices. If $z_0 > 1.16a$ (m = 1), the far field of the system is a dipole field in the first case and a quadrupole field in the second. As m increases the multipolarity of the field increases correspondingly, but in the first case it is still one less than in the second case.

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