ON THE TRANSITION TO TURBULENCE IN MAGNETO-HYDRODYNAMIC FLOWS FOR FINITE DISTURBANCES

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The equations for the turbulent energy of the flow of an electrically conducting fluid in a magnetic field for $R_m \ll 1$ are obtained on the basis of a semiempirical closure of the conservation equations for the second moments of the fluctuating velocities. For Reynolds numbers greater than critical, the energy equation has two positive roots. The larger root corresponds to a regime of development of stationary turbulence, while the smaller determines the threshold of the initial disturbances: If the disturbances are less than the threshold, they die down, while if they are greater, they develop into stationary turbulence (hard excitation). The initial disturbance threshold is strongly enhanced in a magnetic field: It grows as approximately the cube of the local critical Reynolds number.

Linear theories of the stability of MHD flows, as is well known, give vastly overestimated values of the critical Reynolds number as compared to experiment. Detailed experimental investigations on transition to turbulence in MHD flows are lacking. However, it is known in ordinary hydrodynamics that the critical Reynolds number increases with an increasing level of the initial disturbances. Experimental investigations of the flow in a boundary layer [1] have shown that linear stability theory is valid for very low levels of initial turbulence – of order 0.03%. In various technical devices and in the study of the effect of a magnetic field on the critical Reynolds number the flow is turbulent at the entrance of the magnetic field and there are considerable perturbations. It is to be expected that under these conditions nonlinear effects are of considerable importance. A nonlinear semiempirical theory of transition to turbulence, based on balance equations for the second moments of the velocity fluctuations, was proposed in references [2, 3]. A brief exposition of the method and a comparison of the main results and assumptions with experimental data are given herein. An important nonlinear effect, the disturbance threshold for transition to turbulence, is examined.

1. Semiempirical Closure of the Balance Equations for the Second Moments of the Velocity Fluctuations

In the absence of a magnetic field the balance equations for the second moments may be closed [4] with the help of an approximate formula for the energy interchange between the three components of the turbuelnt fluctuations

$$\frac{1}{\rho} \overline{p \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)} = -k \frac{\sqrt{E}}{l} (\overline{u_i u_j} - {}^2/_3 \delta_{ij} E) \qquad (1)$$

and an interpolation formula for the dissipation

$$2\nu \overline{\frac{\partial u_i}{\partial x_\alpha} \frac{\partial u_j}{\partial x_\alpha}} = {}^2/_3 c \delta_{ij} \frac{E^{3/2}}{l} + c_1 \nu \frac{\overline{u_i u_j}}{l^2}, \qquad (2)$$

where $E = {}^1/_2 \overline{u_i u_i}$ is the turbulent energy, l is the scale of the turbulence, and k, c, and c_1 are constants.

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The first term on the right of Eq. (2) is the familiar Kolmogorov formula, which results from the local isotropy hypothesis. The second term describes the direct effect of viscosity on the fluctuations, which gives the main contribution to the Reynolds stresses. This term is essential near the critical regime, while at large Reynolds numbers it is important in the transition region near the wall.

For turbulent flow of an electrically conducting fluid in a magnetic field, terms representing Joule dissipation of the fluctuating energy appear in the balance equations for the second moments [5, 6]. In order to obtain analytic results in subsequent work, we use the simplest expression for the Joule dissipation [3]

$$\left(\frac{\partial \overline{u_i u_j}}{\partial t}\right)_B = -\gamma^2 B^2 \frac{\sigma}{\rho}\overline{u_i u_j}, \tag{3}$$

the meaning of which is particularly clear for $i = j$: In a magnetic field with induction B the velocity fluctuation u induces an electric field of intensity $e \sim uB$ and a current $j \sim \sigma uB$, which produces the Joule heating $\overline{je} \sim \sigma B^2 \overline{u^2}$. The factor $1/\rho$ appears in (3) due to the change to kinematic quantities, while γ may be considered as an empirical coefficient.

Combining the second term in (2) with the expression (3), we have

$$\left(\nu\frac{c_1}{l^2}+\gamma^2 B^2\frac{\sigma}{\rho}\right)\overline{u_i u_j} = \nu\frac{c_1}{l^2}\left(1+\frac{\gamma^2}{c_1}\mathrm{Ha}_l^2\right)\overline{u_i u_j}, \tag{4}$$

where the local Hartmann number is $\mathrm{Ha}_l = Bl\sqrt{\sigma/\mu}$.

Thus in the present approximation the effect of a magnetic field of arbitrary orientation on the turbulence is equivalent to an increase in the dissipation coefficient c_1 by the factor $1+(\gamma^2/c_1)\,\mathrm{Ha}_l^2$. Hence all local relationships derived in [2] may be extended to MHD flow by replacing c_1 by

$$c_{1B} = c_1(1+(\gamma^2/c_1)\,\mathrm{Ha}_l^2). \tag{5}$$

If the molecular viscosity did not appear in the Reynolds equation, the effect of the magnetic field would be equivalent to a concomitant increase of the viscosity (this occurs in the turbulent core of the flow and in the turbulent mixing layer).

In a local treatment $\gamma = \gamma(\mathrm{Ha}_l, \mathrm{R}_l)$, however, we take for simplicity the mean values over the section $\gamma = \gamma(\mathrm{Ha}, \mathrm{R})$. Comparison of the results of calculations of the hydraulic resistance for a flow in a circular tube with experimental data enables the coefficient γ to be determined [7] as a function of Ha and R (Fig. 1). The decrease in γ with Ha is related to the intensified anisotropy of the turbulence. The decrease in γ obtained with R is apparently due to the inexactness of the approximate relations. In subsequent work it is convenient to use the following mean values of the coefficient γ:

$$\gamma \cong 0.3 \quad (\text{for } k/c=7); \quad \gamma \cong 0.25 \quad (\text{for } k/c=1.6). \tag{5'}$$

2. Transition to Turbulence in the Presence of Finite Disturbances

For many types of flows diffusion and convection of turbulent energy are not the governing processes and the energy equation of turbulence* may be represented as [2, 3]

$$^2/_3(k-c)\mathrm{R}_l^2\mathrm{R}_E = (c\mathrm{R}_E + c_{1B})(k\mathrm{R}_E + c_{1B})^2, \tag{6}$$

where the local Reynolds number is $\mathrm{R}_l = (l^2 dU/dy)/\nu$, the turbulent Reynolds number is $\mathrm{R_E} = l\sqrt{E/\nu}$, and c_{1B} is determined by (5).

In investigating the roots of Eq. (6) one may consider that the left side represents production of turbulent energy, and the right side dissipation (Fig. 2a). The roots of the equation are determined by the points of

*In discussing transition it is more correct to say "disturbances" rather than the term "turbulence" used for brevity. The balance equations for the second moments are correct for practically any disturbances. The restrictions applied to the disturbances arise as a result of the semiempirical closure of the equations and are discussed below.

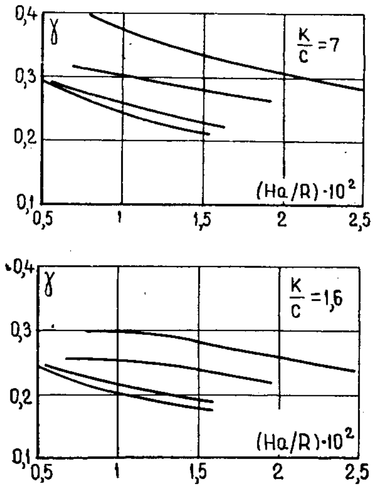


Fig. 1

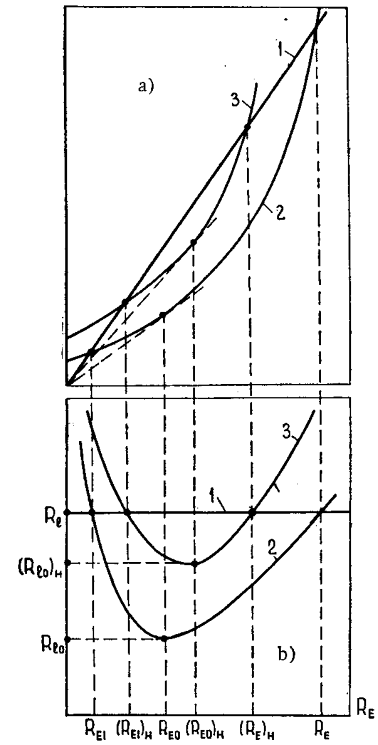


Fig. 2

Fig. 1. Dependence of the coefficient γ on the Hartmann and Reynolds numbers. The curves located higher correspond to larger Reynolds numbers ($R = 1.25 \cdot 10^4$; $R = 3.86 \cdot 10^4$; $R = 7.6 \cdot 10^4$; $R = 9 \cdot 10^4$).

Fig. 2. Geometric interpretation of the turbulent energy balance equation: a) production and dissipation of turbulent energy as a function of R_E: 1 – production, 2 – dissipation for $B = 0$, 3 – dissipation for $B \neq 0$; b) relation of Reynolds numbers R_l and R_E.

intersection of the curves. The relationship between R_l and R_E is illustrated by Fig. 2b. With increasing magnetic field c_{1B} increases, and the curve representing dissipation is located above the curves corresponding to small values of B. The slope of the straight line representing production is proportional to R_l^2. For each value of Ha_l there exists a critical value of the local Reynolds number R_{l0}, for which the curves become tangent. It is clear from Fig. 2 that R_{l0} is the minimum value of R_l, i.e., it is determined by the condition

$$\left(\frac{dR_l}{dR_E}\right)_0 = 0. \tag{7}$$

It can be shown [2, 3] that from (7) and (4) there follows

$$R_{l0} = (R_{l0})_{B=0}\left(1 + \frac{\gamma^2}{c_1}\,Ha_l^2\right). \tag{8}$$

From Fig. 2 it is obvious that for $R_l < R_{l0}$ dissipation is greater than production for all R_E, hence disturbances of any intensity die out.* For $R_l > R_{l0}$ Eq. (6) has two positive roots. The larger root deter-

*The trivial solution $R_E = 0$ corresponding to undisturbed flow is excluded from Eq. (6) as a consequence of cancellation.

mines the stationary turbulent regime. The regime corresponding to the smaller root R_{E1} is unstable. If $R_E > R_{E1}$ for the initial perturbations, then production is larger than dissipation and the disturbances develop to a stationary level (hard excitation). If the initial disturbances have $R_E < R_{E1}$, then production is smaller than dissipation and the disturbances decay. Thus R_{E1} is the threshold of the initial disturbances for transition to turbulence.

The use of the approximate expression (1), according to which energy interchange takes place between the three components of the perturbed velocity in proportion to the differences in energy of these components, puts a restriction on the initial disturbances considered and is the main factor determining the range of applicability of the method. The disturbances may arise, for example, as a result of the inlet design – screen or honeycomb.

The use of the approximate formula (2) for dissipation does not put a significant restriction on the range of applicability of the method, since for small R_E the second term is the dominant one, and this term describes dissipation well for relatively simple forms of disturbances (up to sinusoidal waves).

In a certain energy method [8] disturbances are considered which do not possess the dynamical properties of the real motions and hence vastly underestimate the critical Reynolds number. Approximate relations of the type (1)-(3) are qualitatively correct descriptions of the fundamental properties of the perturbed motion; hence the use of these relations in a semiempirical method should lead to better results than the use of the energy method.

Critical Regime for Flow in a Pipe in a Longitudinal Magnetic Field. The following approximate formulas were obtained in reference [3] for the critical Reynolds number R_0 defining transition to turbulence:

$$R_0 = 28\,\mathrm{Ha}_r + 0.2\,\mathrm{Ha}_r^2 \qquad (k/c = 7);$$
$$R_0 = 36\,\mathrm{Ha}_r + 0.33\,\mathrm{Ha}_r^2 \qquad (k/c = 1.6). \tag{9}$$

These formulas were obtained from expressions of the type

$$R_0 = A\gamma\,\mathrm{Ha} + B\gamma^2\,\mathrm{Ha}^2 \tag{10}$$

for $\gamma^2 = 0.1$. The Hartmann number is based on the radius of the pipe. According to an estimate, the formulas (9) are appropriate for $R_0 > 5000$. Using the estimate of the coefficient γ given above, basing the Hartmann number on the diameter of the pipe, and extending the range of applicability of the formula in the simplest manner up to the critical Reynolds number in the absence of a magnetic field $R_{00} = 2300$, we obtain

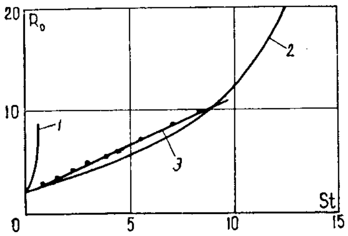
$$R_0 = 2300 + 13\,\mathrm{Ha} + 0.04\,\mathrm{Ha}^2 \qquad (k/c = 7);$$
$$R_0 = 2300 + 14\,\mathrm{Ha} + 0.05\,\mathrm{Ha}^2 \qquad (k/c = 1.6). \tag{11}$$

Thus the coefficients of the critical relation (10) vary little over their whole possible range of variation. Using the average values of the coefficients, we have [7]

$$R_0 = 2300 + 13.5\,\mathrm{Ha} + 0.045\,\mathrm{Ha}^2. \tag{12}$$

Results of an experimental investigation with a hot-wire anemometer of the turbulent velocity fluctuations in a flow of mercury in a pipe with a longitudinal magnetic field were given in [9]. The critical Reynolds number was determined experimentally as a function of the Stuart number St for vanishing of the intermittancy coefficient. For Reynolds numbers up to $R \cong 10^4$ the experimental data is described by the linear relation $R_0/R_{00} = 1 + 0.4$ St (Fig. 3, curve 3). In Fig. 3 are also shown results of Stuart's linear theory (curve 1) and those obtained from Eq. (12) (curve 2). The curves 2 and 3 may be considered satisfactory over the range of the experiments. The main difference between them is that no tendency for intensified growth of R_0 for large St was detected in the experiment. If one considers that γ decreases with Ha, then from (10) it follows that for small St, where the first term is dominant, R_0 increases, but at large St, where the second term is dominant, R_0 is reduced to the value obtained from Eq. (12). Hence for large St the increase of R_0 should be less pronounced than according to (12).

It was established in [9] that in a longitudinal magnetic field, concomitant with the increase in the critical Reynolds number, the range of Reynolds numbers ΔR in which intermittancy occurs at a given Stuart number becomes wider. Analysis of the data presented in [9] shows that the relative width of this interval, $\Delta R/R_0$, decreases with increasing R_0 (Fig. 4).



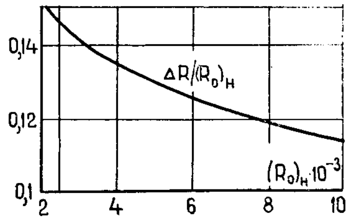


Fig. 3

Fig. 4

Fig. 3. Critical Reynolds number as a function of Stuart number for flow in a tube. 1) Stuart's theory; 2) according to formula (12); 3) experiment [9].

Fig. 4. Relative width of the interval of Reynolds numbers in which intermittancy occurs as a function of the critical Reynolds number for flow in a tube in a longitudinal magnetic field.

Moreover, the oscillograms shown in [9] indicate that the velocity fluctuations in the turbulent "plugs," for flow in a longitudinal magnetic field as well as for B = 0, have a rather developed spectrum up to $R = R_0$.* Both of these facts confirm the assumptions made above concerning the nature of the disturbances and the possibility of an approximate semiempirical description of nonlinear effects in the transition to turbulence.

Threshold of Disturbances for Transition to Turbulence. The disturbance threshold R_{E1} may with Eq. (6) be represented as a function of the local numbers R_l and Ha_l in either a graphical or tabular form. It is clear from Fig. 2b that the threshold R_{E1} decreases with increasing R_l. To obtain simple analytical relations we consider first the case $R_E \ll R_{E0}$, corresponding to the condition $R_l \ll R_{E0}$, which is of interest in practical applications. To estimate the terms of Eq. (6) we use rounded-off values of the quantities [2, 3]: $c \approx 0.2$; $k \approx 0.7$; $c_1 \approx 3$; $(R_{E0})_B/c_{1B} = R_{E0}/c_1 \approx 0.6$; $(R_{l0})_B/c_{1B} = R_{l0}/c_1 > 2.5$. For the first parentheses on the right side of Eq. (6) we have $cR_{E1}/c_1 \ll cR_{E0}/c_1 \approx 0.1$, and for the second, $2kR_{E1}/c_1 \ll 2kR_{E0}/c_1 \approx 1$. Neglecting small terms, we obtain

$$(R_{E1})_B = \frac{c_{1B}^3}{{}^2/_3(k-c)R_l^2} = \frac{c_1^3[1+(\gamma^2/c_1)Ha_l^2]^3}{{}^2/_3(k-c)R_l^2}. \tag{13}$$

In the absence of a magnetic field

$$R_{E1} = c_1^3/[{}^2/_3(k-c)R_l^2]. \tag{14}$$

For the ratio of threshold values R_{E1} in the presence and absence of a magnetic field and for a specified R_l we obtain

$$\frac{(R_{E1})_B}{R_{E1}} = \left(1+\frac{\gamma^2}{c_1}Ha_l^2\right)^3 = \left[\frac{(R_{l0})_B}{R_{l0}}\right]^3 = \left[\frac{(R_{E0})_B}{R_{E0}}\right]^3. \tag{15}$$

Thus the relative increase in the disturbance threshold for transition is equal to the cube of the relative increase of the local critical Reynolds number.

For sufficiently large R_l, and consequently, large Ha_l, we obtain from (13)

$$(R_{E1})_B \cong \frac{(\gamma Ha_l)^6}{{}^2/_3(k-c)R_l^2}. \tag{16}$$

We express the dimensionless numbers in (14) and (16) in terms of the defining quantities

$$\sqrt{E_1} = \frac{c_1^3}{{}^2/_3(k-c)}\,\frac{\nu^3}{l^5(dU/dy)^2}; \quad \sqrt{E_{1B}} = \frac{(\sigma/\rho)^3(\gamma B)^6 l}{{}^2/_3(k-c)\,(dU/dy)^2}. \tag{17), (18}$$

We note that the appearance of the disturbance scale l in the numerator of (18) is a consequence of the small difference of the high powers of l in the numerator and denominator. Noting that γ decreases with Ha_l, we find that $\sqrt{E_{1B}}$ decreases with increasing scale, but apparently considerably less than for B = 0. The

*The value of the experimental data of [9] is somewhat reduced by the fact that the wire of the hot-wire anemometer was stretched over the entire diameter of the channel. In this case the developed fluctuation velocity spectrum may be a result of the superposition of random fluctuations of different scale near and remote from the channel walls.

dependence of the disturbance threshold on the velocity gradient is the same with or without a magnetic field. The larger dU/dy is, i.e., the closer the flow is to a shear layer, the lower is the disturbance threshold.

It follows from (15) that in a magnetic field R_{E1} grows considerably more rapidly that R_{E0}; hence we remove the limitation $R_{E1} \ll R_{E0}$ made earlier. The stability threshold has a meaning for Reynolds numbers larger than critical; hence we have the natural restrictions: $R_l \geq R_{l0}$, $R_{E1} \leq R_{E0}$. We represent (6) in the form

$$^2/_3(k-c)R_l^2(R_{E1})_B = k^2c(R_{E1}^3)_B + (k+2c)kc_{1B}(R_{E1}^2)_B + (2k+c)c_{1B}^2(R_{E1})_B + c_{1B}^3. \qquad (19)$$

We estimate the relative magnitudes of the terms on the right side as follows:

$$\frac{k^2c(R_{E1}^3)_B}{(k+2c)kc_{1B}(R_{E1}^2)_B} \leqslant \frac{kc}{k+2c}\,\frac{(R_{E0})_B}{c_{1B}} = \frac{kc}{k+2c}\,\frac{R_{E0}}{c_1} \approx 0.1;$$

$$\frac{(k+2c)kc_{1B}(R_{E1}^2)_B}{(2k+c)c_{1B}^2(R_{E1})_B} \leqslant \frac{k+2c}{2k+c}\,\frac{(R_{E0})_B}{c_{1B}} \approx 0.3;$$

$$\frac{(2k+c)c_{1B}^2(R_{E1})_B}{c_{1B}^3} \leqslant (2k+c)\frac{(R_{E0})_B}{c_{1B}} \approx 1.$$

Retaining the two last terms on the right side of Eq. (19), we have

$$(R_{E1})_B = \frac{c_{1B}^3}{^2/_3(k-c)R_l^2 - (2k+c)c_{1B}^2}. \qquad (20)$$

For large Ha_l

$$(R_{E1})_B = \frac{(\gamma Ha_l)^6}{^2/_3(k-c)R_l^2 - (2k+c)(\gamma Ha_l)^4}. \qquad (21)$$

Let us estimate the limit of applicability of (13). From the condition

$$\frac{(2k+c)(\gamma Ha_l)^4}{^2/_3(k-c)R_l^2} \ll 1$$

it follows that $S_l^2 \ll 60$, where $S_l = Ha_l^2/R_l$. Thus the simplified formula (13) may be used up to $S_l \approx 3$. For large S_l the formula (13) underestimates the value of $(R_{E1})_B$.

As with any semiempirical theory, the one discussed above is lacking in experimental verification and refinement. In all experimental investigation of flow in a pipe in a longitudinal magnetic field it is of the greatest interest to determine if the intensified growth of the critical Reynolds number at large Stuart numbers predicted by the theory actually occurs. For this it is necessary to expand the range of Stuart numbers studied up to St ≈ 30. Experimental confirmation of the intensified growth of the initial disturbance threshold in a magnetic field will indicate that in MHD flows the transition to turbulence may be delayed until considerably larger Reynolds numbers (or up to considerably larger disturbances of the flow) than in flows without a magnetic field.

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