UNSTEADY SHEAR FLOWS OF A CONDUCTING FLUID WITH A RHEOLOGICAL POWER LAW

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UDC 538.4

Certain classes of unsteady plane shear flows of conducting non-Newtonian fluids with a rheological power law in the presence of a transverse magnetic field, for which exact self-preserving solutions are obtained, were investigated. An effect was found which consisted in that, in the case of dilatant fluids, the shear disturbances propagating with finite velocity prove to be localized in a finite region of space.

In this article we will consider the motions of a conducting incompressible non-Newtonian fluid with a rheological power law for which the relation between the shear stress τ and the velocity gradient $\partial u/\partial z$ in the case of plane motions has the form [1]

$$\tau=k\left|\frac{\partial u}{\partial z}\right|^{n-1}\frac{\partial u}{\partial z}\qquad(n>0). \tag{1}$$

Here k and n are the rheological constants of the medium, the case $n = 1$ corresponding to a Newtonian viscous fluid.

Steady MHD flows of the indicated media were investigated in [2, 3]; as was noted in these investigations, the character of the flows of dilatant fluids ($n > 1$) differs considerably from the character of flows of pseudoplastic fluids ($n < 1$). In particular, quasisolid flow zones in which the fluid moves with a velocity constant over the channel section (magnetic plasticity effect) appear during flows of conducting dilatant fluid in channels for values of the Hartmann number, generalized to the case of these media, exceeding some critical value [2]. This permits the expectation that peculiarities related with the non-Newtonian properties of the fluid can be observed also in unsteady MHD flows of such media.

We will consider certain unsteady shear MHD flows of fluids with a rheological power law (1) in a constant homogeneous external magnetic field.

Let the resting non-Newtonian fluid obeying rheological law (1) and having finite conductivity σ occupy a half-space $-\infty < z < 0$. Here it is assumed that the external magnetic field is directed along the z axis and an electric field is absent in the case being considered, $E = 0$. An arbitrary movement of a plate lying on the fluid ($z = 0$) effects an unsteady flow of the fluid, during which at any time $\partial u/\partial z > 0$. In solving the corresponding problems it is assumed that the law of motion of the plate $u(0, t) = U(t)$ for $t > 0$ or the character of the dependence of the tangential stresses acting on the plate $\tau(0, t) = \tau_w(t)$ is known.

The equation of motion in the case of gradient-free MHD flows with consideration of the comments made above is written in the form

$$\rho\frac{\partial u}{\partial t}=\frac{\partial}{\partial z}\left[k\left(\frac{\partial u}{\partial z}\right)^n\right]-\sigma B_0^2 u, \tag{2}$$

where $u(z, t)$ is the velocity of the fluid flow, ρ is the density of the fluid, and B_0 is the induction of the external transverse magnetic field. Introducing the function $\omega(z, t) \equiv \tau^{1/n} = k^{1/n}\,\partial u/\partial z$ and differentiating

Translated from Magnitnaya Gidrodinamika, No. 2, pp. 50-58, April-June, 1971. Original article submitted October 5, 1970.

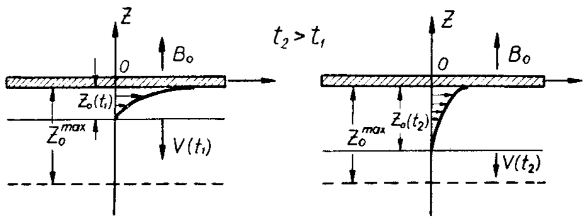

Fig. 1

Eq. (2) with respect to z, we write it in the form

$$\frac{\partial \omega}{\partial t} = a \frac{\partial^2 \omega^n}{\partial z^2} - \lambda \omega, \tag{3}$$

where $a = k^{1/n}/\rho$ and $\lambda = \sigma B_0^2/\rho$.

The solution of Eq. (3) can be represented in the form

$$\omega(z, t) = v(z, t) \exp[-\lambda t]. \tag{4}$$

Substituting (4) into (3), we obtain the equation for determining the function v

$$\frac{\partial v}{\partial \theta} = a \frac{\partial^2 v^n}{\partial z^2} \quad \left(\theta = \frac{1 - \exp[-\lambda(n-1)t]}{\lambda(n-1)}\right). \tag{5}$$

The analytic solutions of nonlinear equation (5) corresponding to the arbitrary law of motion of the plate U(t) or arbitrary dependence $\tau_w(t)$ cannot be found. However, for certain special cases of these dependences exact selfpreserving solutions can be obtained. An analysis of these solutions permits investigating the regularities related with the presence of non-Newtonian properties of the fluid. We will examine three such problems.

1. Let the velocity of the plate vary when $t > 0$ according to the law $U(t) = U_0 \exp[-\lambda t]$. We will find the distribution of shear stresses in the fluid at any time $t > 0$.

The solution of Eq. (5) satisfying the pertinent initial and boundary conditions will be sought in the form

$$v(z, \theta) \equiv v(\eta, \theta) = \left(\frac{W^2}{a\theta}\right)^{\frac{1}{n+1}} \varphi(\eta) \tag{6}$$

with a selfpreserving variable

$$\eta = z \left(\frac{W^{1-n}}{a\theta}\right)^{\frac{1}{n+1}}. \tag{7}$$

[In expressions (6) and (7) the dimensional constant $W = k^{1/n} U_0$.]

Substituting the desired form of the solution (6) into Eq. (5), we obtain an ordinary differential equation for determining the function φ

$$\frac{d}{d\eta}\left[(n+1)\frac{d\varphi^n}{d\eta} + \varphi\eta\right] = 0. \tag{8}$$

The solution of this equation differs considerably for the cases $n > 1$ and $n < 1$.

For dilatant fluids ($n > 1$) the solution of Eq. (8) is

$$\varphi(\eta) = \begin{cases} \left[\dfrac{n-1}{2n(n+1)}(\eta_0^2 - \eta^2)\right]^{\frac{1}{n-1}} & |\eta| < \eta_0 \, . \\ 0 & |\eta| \geqslant \eta_c \end{cases} \tag{9}$$

Here η_0 is a certain constant whose value will be determined below with consideration of the boundary condition for velocity.

This solution corresponds to the circumstance that in dilatant fluids ($n > 1$) at any time $t > 0$ the stresses caused by motion of the plate have time to propagate only in a layer of fluid of finite thickness: $-z_0(t < z < 0$; outside this layer the tangential stresses are equal to zero, and the fluid is at rest (Fig. 1). In other words, in dilatant fluids ($n > 1$) the front of the shear-stress wave propagates with finite velocity V(t).

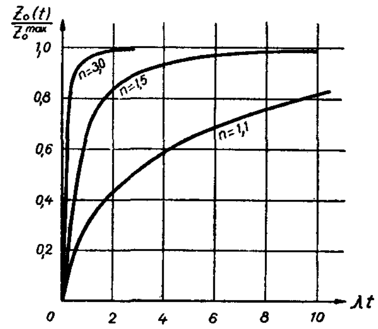

Fig. 2

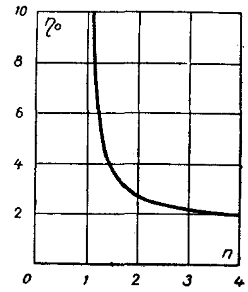

Fig. 3

The position of the front of the shear wave $z_0(t)$ at any time can be found from the condition $|\eta| = \eta_0$, which with consideration of (7) determines the expression

$$z_0(t) = \eta_0 \left[\frac{W^{n-1}a}{\lambda(n-1)}\right]^{\frac{1}{n+1}} \{1-\exp[-\lambda(n-1)t]\}^{\frac{1}{n+1}} \equiv z_0^{\max}\{1-\exp[-\lambda(n-1)t]\}^{\frac{1}{n+1}}. \quad (10)$$

Figure 2 shows the position of the front of the shear wave as a function of time for different values of the rheological constant n.

We need call particular attention to the specific character of the shear flow occurring when $\lambda \neq 0$ and $t \to \infty$. In this case

$$z_0 \to z_0^{\max} = \eta_0 \left[\frac{W^{n-1}a}{\lambda(n-1)}\right]^{\frac{1}{n+1}} = \mathrm{const} < \infty, \quad (11)$$

i.e., for the given law of motion of the plate the shear disturbances do not extend beyond a zone whose width is determined by the quantity z_0^{max} (see Fig. 1).

It is of interest to analyze the effect of the external magnetic field on the depth of penetration z_0^{max} of shear disturbances in a magnetized fluid. If follows from (11) that

$$z_0^{\max} \sim B_0^{-\frac{2}{n+1}}.$$

Thus, the depth of penetration of the shear disturbances decreases with an increase of induction of the external magnetic field. This permits the conclusion that in strong magnetic fields, even in the presence of insignificant deviations from Newtonian properties, the character of flow of the fluid can differ considerably from the flow of a Newtonian fluid.

We note also that when $\lambda \to 0$ and $t \to \infty$ the indicated effect of localization of shear stresses in the absence of a magnetic field disappears ($z_0^{max} \to \infty$ when $\lambda \to 0$).

The velocity of the front of the shear wave is equal to

$$V(t) = \frac{dz_0}{dt} = z_0^{\max}\frac{\lambda(n-1)}{n+1}\{1-\exp[-\lambda(n-1)t]\}^{-\frac{n}{n+1}}\exp[-\lambda(n-1)t]. \quad (12)$$

It is obvious that it depends on the magnitude of induction of the external magnetic field.

The constant η_0 figuring in Eqs. (9)-(12) can be found from the following considerations. By definition of the function ω with consideration of the law of motion of the plate, we have the obvious equality

$$\int_{-z_0(t)}^{0} \omega(z,t)\,dz = k^{1/n}u(0,t) = W\exp[-\lambda t], \quad (13)$$

which, taking into account (4), (6), and (7), we can write in the form

$$\int_0^{\eta_0} \varphi(\eta)\,d\eta = 1. \tag{14}$$

Performing integration in (14) with consideration of expressions (9), we can obtain

$$\eta_0 = \left[\frac{2n(n+1)}{n-1}\right]^{\frac{1}{n+1}} \left[\frac{1}{2} B\left(\frac{1}{2}, \frac{n}{n-1}\right)\right]^{\frac{1-n}{1+n}}, \tag{15}$$

where B (α, β) is the beta function.

Figure 3 presents η_0 as a function of n. As we see from the figure, $\eta_0 \to \infty$ when $n \to 1$. With consideration of (11) and (12) this corresponds to the circumstance that in Newtonian fluids the propagation velocity of shear disturbances is infinite and the effect of spatial localization of the disturbances does not occur. Accordingly, for this case, carrying out passages to the limit $\eta \to 1$ in formulas (6), (7), (9), and (15), we have

$$\begin{gathered} v(z, t) = \frac{W}{\sqrt{\nu t}} \lim_{n \to 1} \left\{ \frac{1}{\sqrt{\pi}} \left[1 - (n-1)\frac{z^2}{4\nu t}\right]^{\frac{1}{n-1}} \right\} = \frac{W}{\sqrt{\pi \nu t}} \exp\left[-\frac{z^2}{4\nu t}\right], \\ \omega(z, t) = \frac{W}{\sqrt{\pi \nu t}} \exp\left[-\frac{z^2}{4\nu t} - \lambda t\right], \\ u(z, t) = \frac{1}{k}\int_{-\infty}^{z} \omega(z, t)\,dz = U_0 \exp[-\lambda t]\left\{1 - \Phi\left(\frac{|z|}{2\sqrt{\nu t}}\right)\right\}, \end{gathered} \tag{16}$$

where $\Phi(y)$ is the error function and $\nu = k/\rho$ is the coefficient of kinematic viscosity of Newtonian fluid.

The other passage to the limit $\lambda \to 0$ corresponds to flow of a nonconducting dilatant fluid, when the motion of the plate when $t > 0$ occurs with a constant velocity U_0. In this case we have from (4), (6), (7), and (9):

$$\omega(z, t) = \begin{cases} \left(\frac{W^2}{at}\right)^{\frac{1}{n+1}} \left[\frac{n-1}{2n(n+1)}(\eta_0^2 - \eta^2)\right]^{\frac{1}{n-1}} & |\eta| < \eta_0, \\ 0 & |\eta| \geq \eta_0, \end{cases} \tag{17}$$

where $\eta = zW^{1-n/1+n}(at)^{-1/n+1}$; the constant η_0 is determined by relation (15). Thus, also in the case of a nonconducting dilatant fluid (or in the absence of an external transverse magnetic field) we observe shear waves whose existence is related with the presence of non-Newtonian properties in the medium, when the effective viscosity of the fluid depends on the velocity gradient and vanishes at a point where $\partial u/\partial z = 0$.

In the case of the motion of a nonconducting dilatant fluid ($\lambda = 0$) the position of the front of the shear wave $z_0(t)$ is determined with consideration of (17) by the relation

$$z_0(t) = \eta_0 W^{\frac{n-1}{n+1}} (at)^{\frac{1}{n+1}} \equiv \text{const} \cdot t^{\frac{1}{n+1}}, \tag{18}$$

and the propagation of the wave front V(t) is equal to

$$V(t) = \eta_0 \frac{W^{\frac{n-1}{n+1}} a^{\frac{1}{n+1}}}{n+1} t^{-\frac{n}{n+1}} \equiv \text{const} \cdot t^{-\frac{n}{n+1}}. \tag{19}$$

For pseudoplastic fluids ($n < 1$) the solution of Eq. (8) corresponding to the statement of the problem is the expression

$$\varphi(\eta)=\left[\frac{1-n}{2n(n+1)}(\eta^2+A^2)\right]^{\frac{1}{n-1}}\quad(|\eta|<\infty);\tag{20}$$

the need to satisfy the boundary condition corresponding to the given law of motion of the plate

$$\int_{-\infty}^{0}\omega(z,t)\,dz=W\exp[-\lambda t]\tag{21}$$

with consideration of (4), (6), and (7) permits determining the constant A from

$$\int_{0}^{\infty}\varphi(\eta)\,d\eta=1.\tag{22}$$

Hence, taking into account expression (20), after calculations we have

$$A=\left[\frac{2n(n+1)}{1-n}\right]^{\frac{1}{n+1}}\left\{\frac{1}{2}B\left[\frac{1}{2},\frac{1+n}{2(1-n)}\right]\right\}^{\frac{1-n}{1+n}}.\tag{23}$$

Solution (20) corresponds to instantaneous propagation of a shear disturbance in pseudoplastic fluids ($n < 1$) unlike dilatant fluids, where, as was shown above, the propagation velocity of the shear waves is finite.

We can also carry out passages to the limit $n \to 1$ and $\lambda \to 0$ for pseudoplastic fluids ($n < 1$). The first of them (passage to a Newtonian fluid), carried out in (20) with consideration of (23), determines, as could be expected, relations (16). The second passage to the limit $\lambda \to 0$ (passage to a nonconducting pseudoplastic fluid) gives

$$\omega(\eta,t)=\left(\frac{W^2}{at}\right)^{\frac{1}{n+1}}\left[\frac{1-n}{2n(n+1)}(\eta^2+A^2)\right]^{\frac{1}{n-1}},\tag{24}$$

where

$$\eta=z\left(\frac{W^{1-n}}{at}\right)^{\frac{1}{n+1}},$$

and constant A is determined by relation (23).

2. The conclusion obtained concerning the possibility of the propagation of plane shear waves in dilatant fluids exhibits a general character and is not related with a certain special time dependence of the velocity of the plate considered in solving the preceding problem. We can be convinced of this by analyzing the solutions of other problems for which it is also possible to find an exact analytic solution. In particular, we can check by direct substitution that for the case $n > 1$ the solution of Eq. (5) is also

$$v(z,\theta)\equiv v(\zeta,\theta)=\begin{cases}v_0(a\theta)^{\frac{1}{n-1}}[1+\zeta]^{\frac{1}{n-1}} & |\zeta|<1,\\ 0 & |\zeta|\geqslant 1,\end{cases}\tag{25}$$

where $\zeta = z/ca\theta$, and constant c is determined from the condition $c^2=\frac{n}{n-1}v_0^{n-1}$.

Solution (25) also corresponds to a plane shear wave, the position of the front $z_0(t)$ of which is determined by the condition $|\zeta| = 1$, i.e.,

$$z_0(t)=ca\theta\equiv ca\left[\frac{1-\exp[-\lambda(n-1)t]}{\lambda(n-1)}\right].\tag{26}$$

As we see from (26), in this problem also there occurs the effect of spatial localization of shear disturbances, whereat $z_0^{max} = ca/\lambda\,(n-1)$.

The propagation velocity of the wave front in this case is equal to

$$V(t) = ca\exp[-\lambda(n-1)t]. \tag{27}$$

To provide such a flow regime the stress on the plate τ_w must change according to the law

$$\tau_w(t) = v_0{}^n a^{\frac{n}{n-1}}\left[\frac{1-\exp[-\lambda(n-1)t]}{\lambda(n-1)}\right]^{\frac{n}{n-1}}\exp[-n\lambda t]. \tag{28}$$

We note that this case is interesting in that on passing to a nonconducting dilatant fluid ($\lambda \to 0$) it corresponds to the motion of the front of a shear wave with a constant velocity. Actually, it follows here from (25) that

$$v(z, t) = \begin{cases} v_0 a^{\frac{1}{n-1}}\left(t+\dfrac{z}{ca}\right)^{\frac{1}{n-1}} & |z| < cat, \\ 0 & |z| \geqslant cat, \end{cases} \tag{29}$$

which corresponds to a plane shear wave propagating in a dilatant fluid with a constant velocity $V_0 = ca =$ const. This conclusion can be obtained directly from (27) by passage to the limit.

3. We will consider still another problem of unsteady MHD shear flow of a fluid with a rheological power law having an exact selfpreserving solution.

Let the fluid at rest occupy a half-space $-\infty < z < 0$. The weightless plate lying on the fluid ($z = 0$) as a result of instantaneous shear at time $t = 0$ transmits to the fluid an impulse whose magnitude, referred to a unit area of the plate, is equal to P_0. We will find the velocity distribution in the fluid at any time $t > 0$.

In this case we will seek the solution of Eq. (2) in the form

$$\rho u = \left[\frac{P_0{}^{n+1}}{a^n\theta(t)}\right]^{\frac{1}{2n}} f(\xi)\exp[-\lambda t], \tag{30}$$

where

$$\xi = z\left(\frac{P_0{}^{1-n}}{a^n\theta}\right)^{\frac{1}{2n}} \qquad (-\infty < \xi \leqslant 0),$$

and the other notations are identical to those adopted in solving the preceding problems.

Substituting (30) into (2), we can obtain an equation determining the function f:

$$\frac{d}{d\xi}\left[\left(\frac{df}{d\xi}\right)^n + \frac{1}{2n}f\xi\right] = 0. \tag{31}$$

For dilatant fluids ($n > 1$) Eq. (31) with consideration of the boundary and initial conditions has the form of a shear-impulse wave

$$f(\xi) = \begin{cases} \left[\left(\dfrac{1}{2n}\right)^{\frac{1}{n}}\dfrac{n-1}{n+1}\left(\xi_0{}^{\frac{n+1}{n}} - |\xi|^{\frac{n+1}{n}}\right)\right]^{\frac{n}{n-1}} & |\xi| < \xi_0, \\ 0 & |\xi| \geqslant \xi_0, \end{cases} \tag{32}$$

where $\xi_0 > 0$ is a constant, the value of which will be determined below.

It follows from (32) that the position of the front of the shear wave is determined by the relation

$$z_0(t) = \xi_0(P_0{}^{n-1}a^n\theta)^{\frac{1}{2n}} \equiv z_0^{max}\{1-\exp[-\lambda(n-1)t]\}^{\frac{1}{2n}}, \tag{33}$$

and the size of the region of penetration of the shear disturbance from the plate into the magnetized fluid z_0^{max} depends on the parameters of the problem in the following manner:

$$z_0^{\max}=\xi_0\left[\frac{P_0^{n-1}a^n}{\lambda(n-1)}\right]^{\frac{1}{2n}}. \tag{34}$$

We will find the value of the constant ξ_0. For this purpose we integrate (2) with respect to z from $-z_0(t)$ to 0. With consideration of (32) we obtain:

$$\frac{dP}{dt}=-\lambda P;\quad P(t)=\int\limits_{-z_0(t)}^{0}\rho u(z,t)\,dz. \tag{35}$$

Since in accordance with the condition of the problem $P(0) = P_0$, it follows from (35) that the total impulse of the conducting fluid with a rheological power law moving in a transverse magnetic field varies with time according to the law

$$P(t)=P_0\exp[-\lambda t].$$

This permits determining now the constant ξ_0 from the condition

$$\int\limits_{-z_0(t)}^{0}\rho u(z,t)\,dz=P_0\exp[-\lambda t], \tag{36}$$

which with consideration of (30) and (32) can be transformed to the form

$$\int\limits_{0}^{\xi_0} f(\xi)\,d\xi=1. \tag{37}$$

Calculating the quadrature in (37), with consideration of (32) we obtain

$$\xi_0=(2n)^{\frac{1}{2n}}\sqrt{\frac{n+1}{n-1}}\left[\frac{n}{n+1}\,\mathrm{B}\left(\frac{n}{n+1},\ \frac{2n-1}{n-1}\right)\right]^{\frac{1-n}{2n}} \tag{38}$$

For pseudoplastic fluids in $(n < 1)$ the solution of Eq. (31)

$$f(\xi)=\left[\left(\frac{1}{2n}\right)^{\frac{1}{n}}\frac{1-n}{1+n}\left(|\xi|^{\frac{n+1}{n}}+C^{\frac{n+1}{n}}\right)\right]^{\frac{n}{n-1}}\qquad(|\xi|<\infty),$$

$$C=(2n)^{\frac{}{2n}}\sqrt{\frac{1+n}{1-n}}\left[\frac{n}{n+1}\,\mathrm{B}\left(\frac{n}{n+1},\ \frac{2n^2}{1-n^2}\right)\right]^{\frac{1-n}{2n}} \tag{39}$$

corresponds to the instantaneous involvement of all layers of the fluid into motion, i.e., to an infinite value of the propagation velocity of the shear disturbances.

The formulas obtained above permit also in this problem carrying out passages to the limit $\lambda \to 0$ and $n \to 1\pm 0$. Without presenting the corresponding expression, we note that, as their analysis shows, in the case of a nonconducting dilatant fluid $(n > 1,\ \lambda = 0)$ shear disturbances propagate from the plate with a finite velocity, but their depth of penetration into the medium is infinite ($z_0^{max} \to \infty$ when $\lambda \to 0$).

4. Thus, the following effects occur in conducting dilatant fluids in the presence of a transverse magnetic field: 1) effect of finite propagation velocity of shear disturbances; 2) effect of spatial location of shear disturbances.

The first of these effects is not related ultimately with the effect of the external magnetic field and is observed also in nonconducting dilatant fluids. The presence of this effect is explained physically by the fact that the effective viscosity of dilatant fluids depends on the velocity gradient and vanishes at the front of the shear wave. This fact was noted also in [4], where by means of an expansion in a small parameter the problem of unsteady motion of a conducting fluid with a rheological power in a magnetic field varying in time according to a prescribed (in conformity with the selection of the selfpreserving variable) law was solved. The presence of the effect of a finite propagation velocity of shear disturbances in dilatant fluids could be expected on the basis of the results of [5, 6] in which selfpreserving solutions of quasilinear parabolic equations of type (5) were investigated with respect to problems of the theory of thermal conductivity and percolation.

The second effect – the effect of spatial localization of shear disturbances, which occurs only in conducting media and is related with the effect of the magnetic field – is qualitatively new. This effect explains the appearance of quasisolid zones in steady flows of conducting dilatant fluids in MHD channels [2].

The detection of the effect of spatial localization of shear disturbances in a conducting dilatant fluid indicates the possibility of analogous effects in the theory of nonlinear thermal conductivity. Actually, we can show that the effect of spatial localization of thermal disturbances occurs in media with a coefficient of thermal conductivity dependent on temperature according to the power law in the presence of heat "sinks," the power of which is porportional to the temperature.

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