DISTRIBUTION OF CURRENT DENSITY IN THE CHANNEL OF AN ALTERNATING-CURRENT CONDUCTION MACHINE

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The problem of the flow of an electrically conducting fluid in an infinitely long channel with one pair of electrodes located in a magnetic field varying in time according to the cosine law is considered. An analytical description is obtained for the current density under the assumption that the distribution of the hydrodynamic quantities is given. It is shown that the solution consists of a part analogous to boundary value problems for a constant field and an additional part which depends on the parameters of the alternating field.

We will consider the problem of a flow of an electrically conducting fluid with constant conductivity σ in a time-alternating magnetic field in an infinitely long channel with plane walls $y = \pm h$; a part of the walls $|x| \leq \lambda$ represents symmetric, perfectly conducting electrodes, and the remaining part, insulators (Fig. 1).

The time-alternating magnetic field is perpendicular to the plane of the flow and depends only on ordinate x, $\mathbf{B}(0, 0, B(x, t))$. Assuming that the height of the channel d is less than the effective depth of penetration of the field, we can consider that all parameters of the process are constant along the z axis and we can consider the plane-parallel problem.

The expediency of formulating the problem being considered increases in connection with the fact that ac conduction liquid-metal machines (for example, electromagnetic single-phase pumps) are presently being designed and manufactured. At the same time many theoretical problems for such machines (calculation of spatial effects, end losses, etc.) have actually not been elaborated, unlike the corresponding dc devices. This is explained partially by the circumstance that in an ac machine a quasistationary approximation is impossible, i.e., $\mathrm{rot}\,\mathbf{E} \neq 0$ [1], which complicates the problem.

The alternating fields $\mathbf{j}$, $\mathbf{E}$ in the flow region are found from a system of equations

$$\mathrm{rot}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \qquad \mathrm{div}\,\mathbf{j} = 0 \tag{1}$$

and Ohm's law

$$\mathbf{j} = \sigma[\mathbf{E} + \mathbf{V} \times \mathbf{B}]. \tag{2}$$

It is assumed further that the resultant magnetic field in the fluid is close to the external. This is approximately true in the case of compensation of the "armature reaction" (by a compensating bus bar) or in the case of a small magnetic Reynolds number $\mathrm{Re}_m = \mu\sigma VL$.

Thus the resultant field is assumed to be known, created only by the exciting winding [1] located outside the zone of the channel; therefore $\mathrm{rot}\,\mathbf{B} = 0$ is assumed in the channel zone.

The set of indicated equations permits determining j for given boundary conditions and parameters of the external circuits.

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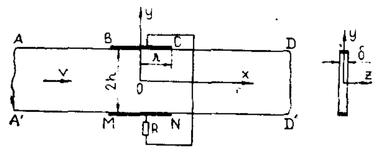

Fig. 1

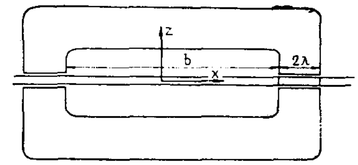

Fig. 2

We introduce the scalar and vector potentials

$$\mathbf{E}=-\operatorname{grad}\varphi-\frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B}=\operatorname{rot}\mathbf{A}. \tag{3}$$

Applying the operation div to Eq. (2), with consideration of (3) and the introduced assumption $\operatorname{rot}\mathbf{B}=0$, we obtain

$$\Delta\varphi+\frac{\partial}{\partial t}(\operatorname{div}\mathbf{A})-\mathbf{B}\operatorname{rot}\mathbf{V}=0. \tag{4}$$

Having used gauge invariance of fields **E** and **B**, we impose on the vector potential **A** the additional condition [2]

$$\operatorname{div}\mathbf{A}=0. \tag{5}$$

Assuming movement of the electrically conducting fluid only longitudinally V(0, 0, V) and setting V = const, from Eq. (4) with consideration of (5), we obtain

$$\Delta\varphi=0. \tag{6}$$

Ohm's law in the projections with consideration of (3) and of the assumption made is written in the form

$$j_x=-\sigma\left(\frac{\partial\varphi}{\partial x}+\frac{\partial A_x}{\partial t}\right); \qquad j_y=-\sigma\left(\frac{\partial\varphi}{\partial y}+\frac{\partial A_y}{\partial t}\right)-\sigma VB. \tag{7}$$

We will further consider everywhere that the magnetic field depends on the space coordinate and time in the following way: $B(x, t)=B(x)\cos\omega t$, whereby the field is concentrated only at the electrodes and, outside of them, is equal to zero:

$$B(x)=\begin{cases}-B_0=\text{const} & \text{for } |x|\leqslant\lambda,\\ 0 & \text{for } |x|>\lambda.\end{cases} \tag{8}$$

Then it is permissible to select A in the form

$$\mathbf{A}(0,\ A_y(x,\ t),\ 0),$$

where $A_y(x, t)=A_y(x)\cos\omega t$,

$$A_y(x)=\begin{cases}-B_0x & \text{for } |x|\leqslant\lambda,\\ B_0\lambda & \text{for } x<-\lambda,\\ -B_0\lambda & \text{for } x>\lambda.\end{cases} \tag{9}$$

Selection of A in form (9) satisfies the equations $\mathbf{B}=\operatorname{rot}A$ and $\operatorname{div}\mathbf{B}=0$ with consideration of (8).

By the quantities φ, A_y, and B we must understand everywhere certain quantities $\varphi(x, y, z, t)$, $A_y(x, z, t)$, $B(x, z, t)$ averaged over z [1].

The independence of B and A_y from coordinate y is possible in view of the fact that the dimensions of the magnetic system with respect to this coordinate are assumed to be quite appreciable. At the same time, the dimension of the channel along coordinate y is limited and equal to 2h. At distance $b/2\gg2\lambda$ (Fig. 2) the currents practically die out, and therefore we can examine only the right half of the magnetic system, considering the channel to be infinitely extended. Such a variant of the flow in a channel in a real magnetic system should be described sufficiently accurately by the formulation of the problem presented.

We will consider the boundary conditions. A transverse current is absent on the insulator walls, $j_y = 0$, and therefore from (7) we obtain

$$\frac{\partial\varphi}{\partial y}=-\frac{\partial A_y}{\partial t}-VB \quad \text{for} \quad |x|>\lambda, \quad y=\pm h. \tag{10}$$

The condition $E_x = 0$ should be fulfilled on the ideal conductor walls, i.e., from (7):

$$\partial\varphi/\partial x=0 \quad \text{for} \quad |x|\leqslant\lambda, \quad y=\pm h. \tag{11}$$

When $x \to \pm\infty$ the interaction of the medium and magnetic field ceases, and therefore the current densities j_x and j_y approach zero:

$$\mathbf{j}(x\to\pm\infty, y)=0. \tag{12}$$

The boundary conditions must be supplemented by an equation for the external circuit connected to the electrodes. Ohm's law for the external circuit consisting, for example, only of an active load, is written as is conventional in low-frequency electrical engineering, the field E in the external circuit being considered the potential field; i.e., $\partial A/\partial t = 0$ is assumed in the region where the external circuit is located [3]. In this case the shunt voltage in the external circuit on one hand is equal to the product of the load current and resistance and on the other to the difference of the scalar potentials at the electrodes:

$$Rd\int_{-\lambda}^{\lambda} j_y(x, h, t)\,\mathrm{d}x=\varphi_{BC}(t)-\varphi_{MN}(t). \tag{13}$$

Here R is the resistance of the external circuit, φ_{BC} and φ_{MN} are the values of the scalar potentials at the electrodes.

Thus, it is necessary to solve Eq. (6) with conditions (10)-(13).

We will seek $\varphi(x, y, t)$ in the form

$$\varphi(x, y, t)=\varphi_1(x, y)\cos\omega t+\varphi_2(x, y)\sin\omega t. \tag{14}$$

Then Ohm's law (7) can be written in the form

$$j_x=j_{1x}\cos\omega t+j_{2x}\sin\omega t; \quad j_y=j_{1y}\cos\omega t+j_{2y}\sin\omega t, \tag{15}$$

where $j_{1x}=-\sigma\frac{\partial\varphi_1}{\partial x}$; $j_{2x}=-\sigma\frac{\partial\varphi_2}{\partial x}$; $j_{1y}=-\sigma\left(\frac{\partial\varphi_1}{\partial y}+VB(x)\right)$; $j_{2y}=-\sigma\left(\frac{\partial\varphi_2}{\partial y}-\omega A_y(x)\right)$.

With consideration of (14) Eq. (6) is broken down into two equations for $\varphi_1(x, y)$ and $\varphi_2(x, y)$, each with their own boundary conditions.

For $\varphi_1(x, y)$ from (6) and (10)-(12) we obtain the equation

$$\Delta\varphi_1=0$$

and the boundary conditions

$$\frac{\partial\varphi_1}{\partial x}=0 \quad \text{for} \quad |x|\leqslant\lambda, \quad y=\pm h; \qquad \frac{\partial\varphi_1}{\partial y}=0 \quad \text{for} \quad |x|>\lambda, \quad y=\pm h;$$

$$\frac{\partial\varphi_1}{\partial x}=\frac{\partial\varphi_1}{\partial y}=0 \quad \text{for} \quad x\to\pm\infty. \tag{16}$$

From Eq. (13) with consideration of (14) and (15) we obtain the condition

$$Rd\int_{-\lambda}^{\lambda} j_{1y}(x, h)\,\mathrm{d}x=\varphi_{1BC}-\varphi_{1MN}, \tag{17}$$

where φ_{1BC} and φ_{1MN} are the values of $\varphi_1(x, y)$ at the electrodes. Analogously for $\varphi_2(x, y)$ from (6) and

(10)-(12) we obtain the equation

$$\Delta\varphi_2=0$$

and the boundary conditions

$$\frac{\partial\varphi_2}{\partial x}=0 \quad \text{for} \quad |x|\leqslant\lambda, \quad y=\pm h;$$
$$\frac{\partial\varphi_2}{\partial y}=-Q \quad \text{for} \quad x>\lambda, \quad y=\pm h; \quad \frac{\partial\varphi_2}{\partial y}=Q \quad \text{for} \quad x<-\lambda, \quad y=\pm h;$$
$$\frac{\partial\varphi_2}{\partial x}=0; \quad \frac{\partial\varphi_2}{\partial y}=\mp Q \quad \text{for} \quad x\to\pm\infty, \tag{18}$$

where $Q = \omega B_0 \lambda$.

Analogous to (17) from Eq. (13) follows

$$Rd\int_{-\lambda}^{\lambda} j_{2y}(x, h)\,\mathrm{d}x=\varphi_{2BC}-\varphi_{2MN}, \tag{19}$$

where φ_{2BC} and φ_{2MN} are the values of $\varphi_2(x, y)$ at the electrodes.

The boundary value problems (16) and (18) are solved independently of one another. Either only $\varphi_1(x, y)$ or $\varphi_2(x, y)$ figures in each of them. The distribution of the currents is determined from superposition of the solution of boundary value problems (16) and (18). Boundary value problem (16) describes the motion of the electrically conducting fluid in a two-electrode channel in a constant magnetic field B_0 concentrated only at the electrodes at flow velocity V. The solution of this problem has the form [4]

$$\varphi_1(x, y)=\Re\mathrm{e}\, W(z),$$

where

$$W(\tau')=-\frac{R\sigma G}{(2+R\sigma\alpha)K(k)}\int_0^{\tau'}\frac{\mathrm{d}\varrho}{\sqrt{(1-\varrho^2)(1-k^2\varrho^2)}}. \tag{20}$$

Here $\tau'=ie^{(z+\lambda)\pi/2h}$; $z=x+iy$; $k=e^{-\lambda\pi/h}$; $G=2\lambda VB_0$; $\alpha=K(k')/K(k)$; $k'^2=1-k^2$; K(k) is the complete elliptic integral of the first kind.

We will proceed to the solution of Eq. (18). We note that if we set $V=0$ in the initial equations, we obtain exactly the boundary value problem (18) describing the distribution of currents in a two-electrode channel with a stationary electrically conducting fluid in a magnetic field alternating in time. This problem can be solved by the Keldysh–Sedov formula. At first we introduce the function

$$f(z)=\frac{\partial\varphi_2}{\partial x}-i\left(\frac{\partial\varphi_2}{\partial y}+Q\right)=u(x, y)+iv(x, y),$$

where

$$u(x, y)=\frac{\partial\varphi_2}{\partial x}, \qquad v(x, y)=-\left(\frac{\partial\varphi_2}{\partial y}+Q\right);$$

$f(z)$ is an analytic function, since $\varphi_2(x, y)$ satisfies the Laplace equation, and $Q=\text{const}$.

Thus, according to (18) we must find function $f(z)$ analytic in region $-\infty<\Re\mathrm{e}\,z<\infty, -h\leqslant\Im\mathrm{m}\,z\leqslant h$ satisfying the boundary conditions

$$\begin{aligned} &u=0 && \text{for } |x|\leqslant\lambda, && y=\pm h;\\ &v=-2Q && \text{for } x<-\lambda, && y=\pm h;\\ &v=0 && \text{for } x>\lambda, && y=\pm h;\\ &f(x\to\infty, y)=0, && f(x\to-\infty, y)=-2iQ. \end{aligned} \tag{21}$$

Equation (19) with consideration of Ohm's law (15) will take the form (integration contour ND'DC is used)

$$\int_{\lambda}^{\infty} u(x,\ -h)\,\mathrm{d}x+\int_{\infty}^{\lambda} u(x,\ h)\,\mathrm{d}x-2Qh=Rd\sigma\left[\int_{-\lambda}^{\lambda} v(x,\ h)\,\mathrm{d}x+2\lambda Q\right]. \tag{22}$$

To solve the problem we map the region $-h\leqslant \Im m\, z\leqslant h$ conformally onto the upper half-plane (Fig. 3):

$$\tau=\xi+i\eta=ie^{(z-\lambda)\pi/2h}.$$

The correspondence of the boundaries in this case is established in the following way:

$$\begin{aligned}&\xi=k^{1/2}e^{x\pi/2h} \quad \text{for} \quad y=-h, \quad -\infty<x<\infty;\\ &\xi=-k^{1/2}e^{x\pi/2h} \quad \text{for} \quad y=h, \quad -\infty<x<\infty.\end{aligned} \tag{23}$$

Here the analytic function $f(z)$ changes to the function

$$F(\tau)=f[z(\tau)]=u_1(\xi,\ \eta)+iv_1(\xi,\ \eta).$$

For function $F(\tau)$ from (21) we obtain the boundary conditions

$$\begin{aligned}&u_1=0 \quad \text{for} \quad k\leqslant|\xi|\leqslant 1, \quad \eta=0;\\ &v_1=-2Q \quad \text{for} \quad |\xi|<k, \quad \eta=0;\\ &v_1=0 \quad \text{for} \quad |\xi|>1, \quad \eta=0;\\ &F(\infty)=0, \qquad F(0)=-2iQ.\end{aligned} \tag{24}$$

With consideration of (23) condition (22) takes the form

$$\frac{2h}{\pi}\left[\int_{1}^{\infty} u_1(\xi,\ 0)\,\frac{\mathrm{d}\xi}{\xi}+\int_{-\infty}^{-1} u_1(\xi,\ 0)\,\frac{\mathrm{d}\xi}{\xi}\right]-2hQ=Rd\sigma\left[\frac{2h}{\pi}\int_{-k}^{-1} v_1(\xi,\ 0)\,\frac{\mathrm{d}\xi}{\xi}+2\lambda Q\right]. \tag{25}$$

To find the function analytic in the upper half-plane satisfying boundary conditions (24), we use the Keldysh–Sedov formula [5]

$$F(\tau)=\frac{-2Q}{\pi g(\tau)}\int_{-k}^{k}\frac{g(\varrho)}{\varrho-\tau}\,\mathrm{d}\varrho+\frac{\gamma_0+\gamma_1\tau}{g_1(\tau)}, \tag{26}$$

where

$$g(\tau)=\sqrt{\frac{(\tau+k)(\tau-1)}{(\tau-k)(\tau+1)}}, \qquad g_1(\tau)=\sqrt{(\tau^2-k^2)(\tau^2-1)},$$

and γ_0 and γ_1 are real arbitrary constants subject to determination.

We must take that branch of the radicals which is positive when $\xi>1$. To find the real constant γ_0 we use the last condition of (24). The first term in $F(\tau)$ represents the Cauchy integral. Finding the limiting value of the Cauchy integral at zero and taking into account that after selecting the appropriate branches of the radicals $g(0)=1$ and $g_1(0)=-k$, we obtain

$$\gamma_0=-\frac{4Q}{\pi}(1-k)kK(k). \tag{27}$$

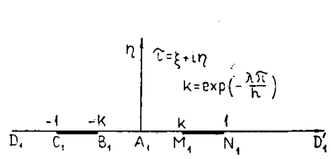

Fig. 3

From condition (25) we find γ_1, having substituted into the integrals the appropriate branches of the functions $g(\tau)$ and $g_1(\tau)$:

$$\gamma_1=\frac{4Qh(H_3+Rd\sigma H_1)+\pi hRd\sigma\gamma_0H_2+\pi^2hQ+\pi^2R\sigma d\lambda Q}{\pi h(Rd\sigma\alpha+2)K(k)}, \tag{28}$$

where

$$H_1=\int_k^1 \sqrt{\frac{(\xi+k)(1-\xi)}{(\xi-k)(1+\xi)}}\frac{d\xi}{\xi}\int_0^k \frac{(\varrho^2-k\varrho^2-k\xi+\varrho^2\xi)\,d\varrho}{(\varrho^2-\xi^2)\sqrt{(k^2-\varrho^2)(1-\varrho^2)}};$$

$$H_2=\int_k^1 \frac{d\xi}{\xi\sqrt{(\xi^2-k^2)(1-\xi^2)}}=\frac{\pi}{2k};$$

$$H_3=\int_1^\infty \sqrt{\frac{(\xi-k)(\xi+1)}{(\xi+k)(\xi-1)}}\frac{d\xi}{\xi}\int_0^k \frac{(\varrho^2-k\varrho^2+k\xi-\varrho^2\xi)}{(\varrho^2-\xi^2)\sqrt{(k^2-\varrho^2)(1-\varrho^2)}}\,d\varrho-$$

$$-\int_1^\infty \sqrt{\frac{(\xi+k)(\xi-1)}{(\xi-k)(\xi+1)}}\frac{d\xi}{\xi}\int_0^k \frac{(\varrho^2-k\varrho^2-k\xi+\varrho^2\xi)}{(\varrho^2-\xi^2)\sqrt{(k^2-\varrho^2)(1-\varrho^2)}}\,d\varrho.$$

H_1, H_2, H_3 can be expressed by elementary functions and elliptic integrals. The equations (20) and (26) obtained with consideration of (27) and (28) permit finding the distribution of currents and the energy characteristics of the channel.

With consideration that

$$\frac{\partial W}{\partial z}=\frac{\partial \varphi_1}{\partial x}-i\frac{\partial \varphi_1}{\partial y},$$

the distribution of the current density in the channel has the form

$$j_x=-\sigma\left[\Re\mathrm{e}\,\frac{\partial W[\tau'(z)]}{\partial z}\cos\omega t+\Re\mathrm{e}\,F[\tau(z)]\sin\omega t\right];$$

$$j_y=\sigma\left\{\left[\Im\mathrm{m}\,\frac{\partial W[\tau'(z)]}{\partial z}-VB(x)\right]\cos\omega t+[\Im\mathrm{m}\,F[\tau(z)]+\omega A_y(x)+Q]\sin\omega t\right\}. \quad (29)$$

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