ROTATION OF AN ELECTRICALLY CONDUCTING FLUID WITH A FREE SURFACE IN A ROTATING FIELD

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The rotation of a thin layer of an electrically conducting fluid partially filling a channel of annular cross section inside a stator with a rotating field is considered with neglect of friction at the channel end. For a turbulent flow, when the fluid velocity is much smaller than the field velocity, the shape of the free surface, the velocity profile, and the losses in the electrically conducting fluid are shown to be dependent on the magnitude of the rotating field with allowance for the transverse edge effect. The calculation is compared with experiment.

We consider the rotation of an electrically conducting fluid 2 (Fig. 1) which partially fills a closed annular vessel formed by the internal surface of the stator of the rotating field 1, the lateral surface of the interior magnetic circuit 3, and the nonconducting end rings 4. Let the force of gravity be in the opposite direction to the y axis, and let the thickness δ, the height $2a$, and the radius R of the channel satisfy the conditions

$$\delta \ll R; \quad \delta \ll 2a. \tag{1}$$

These relations are valid, for example, for the devices described in [1].

When the stator is switched on, the fluid begins to rotate and as a result of the action of the centrifugal forces and the force of gravity, its free surface assumes the shape of the surface of a body of revolution so that in the cross section in Fig. 1 the thickness Δ is not constant along the y coordinate.

The aim of the present paper is to determine the shape of the free surface, the velocity distribution of the rotation, and the losses in the fluid as a function of the magnitude of the rotating field. A similar problem was solved in [2] for a laminar flow with the transverse end effect, which is important in practice, neglected.

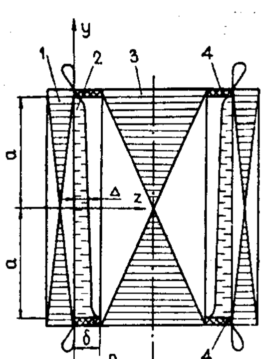

Fig. 1

In the present paper, condition (1) limits the problem to the case of a thin layer of the conducting fluid, which allows us to employ the method developed in [3-5] with averaging of the equations over the small thickness of the layer of the rotating fluid. Therefore, an approximate solution can be obtained for a turbulent flow with allowance for the transverse edge effect by assuming that the rotational velocity of the fluid is much smaller than the field velocity.

1. Derivation of the System of Equations

The relations (1) allow us to consider only the z component of the external H and the induced h magnetic fields and to employ equations with the form and notation of those of [5, 6]:

$$\frac{d^2\dot{h}}{dy^2} - (\alpha^2 + i\mu_0\omega s\sigma)\dot{h} = i\omega s\mu_0\sigma H_m, \tag{2}$$

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where $\dot{h}$ is the complex amplitude of the induced field, and H_m is the amplitude of the external field.

Averaging (2) over z with the aid of $\frac{1}{\delta}\int\limits_0^\delta (\ )dz$ and considering that the electrical conductivity is zero outside the fluid, we obtain

$$\frac{d^2\dot{h}}{dy^2}-\left(\alpha^2+i\mu_0\omega s\sigma\frac{\Delta}{\delta}\right)\dot{h}=i\omega s\mu_0\sigma H_m\frac{\Delta}{\delta}. \tag{3}$$

Equation (3) differs from a similar one in [6] in that Δ is variable along y.

The equation of motion for a turbulent gradientless flow can be written in a form identical to [5]

$$\mu_0\alpha H_m h_a+\lambda\varrho V^2/4\delta=0, \tag{4}$$

where $h_a=\mathfrak{Im}\dot{h}$, λ is the friction coefficient, ρ is the density, and V is the rotational velocity of the fluid averaged over z. As in [5], we assume that λ is constant along y.

The equation for the shape of the free surface can be obtained by using the condition for constant pressure at this surface

$$\frac{d\Delta}{dy}\frac{\partial p}{\partial z}+\frac{\partial p}{\partial y}=0. \tag{5}$$

The conditions (1) allow us to neglect secondary flows and assume zero velocity components of the fluid along the z and y axes. Constructing an equilibrium equation for the z axis, we obtain

$$\partial p/\partial z=-\varrho V^2/R,$$

where the centrifugal body force is on the right-hand side. Constructing an equilibrium equation along the y axis, we obtain

$$\partial p/\partial y=f_y-\varrho g,$$

and f_y is the electromagnetic body force obtained from the x-th density component of the induced currents [5], and g is the acceleration due to the force of gravity.

Taking the expressions for f_y from [5], we obtain instead of (5)

$$\frac{d\Delta}{dy}=-\frac{\mu_0 R}{2\varrho V^2}\left[(H_m+h_r)\frac{dh_r}{dy}+h_a\frac{dh_a}{dy}\right]\frac{\delta}{\Delta}-\frac{gR}{V^2}, \tag{6}$$

where $h_r=\mathfrak{Re}\dot{h}$, and the factor δ/Δ is introduced for averaging of h over the gap δ.

In order to obtain the shape of the free surface as a function of the field strength for a constant fluid volume W_{fl}, we introduce the dependence of the variable volume W on the thickness Δ. With allowance for (1), we obtain

$$\frac{dW}{dy}=2\pi R\Delta. \tag{7}$$

We separate the real and imaginary parts of Eq. (3), first expressing the slip s in terms of the field velocity V_{fi}, and we reduce Eqs. (3), (4), (6), and (7) to dimensionless form by introducing the dimensionless variables

$$\bar{h}_a=h_a/H_m;\quad \bar{h}_r=h_r/H_m;\quad \bar{y}=y\pi/\tau;\quad \bar{\Delta}=\Delta/\delta;\quad \bar{V}=V/V_{fi};\quad \bar{W}=W/2\tau R\delta$$

and the dimensionless parameters

$$\mathrm{Re}_m=\mu_0\sigma V_{fi}\tau/\pi;\quad A=H_m^2\mu_0 R/2\varrho V_{fi}^2\delta;\ \mathrm{Ha}^2=4\mu_0^2H_m^2\delta\sigma/\varrho V_{fi}\lambda;\quad \mathrm{Fr}=V_{fi}^2\delta\alpha/gR.$$

Here, Re_m, A, and Fr are the magnetic Reynolds number, the Alfvén number, and the Froude number, respectively. The parameter Ha differs from the Hartmann number in that the friction per unit volume is averaged for the turbulent flow.

As a result, we obtain a system of dimensionless equations in which, as everywhere in the following text, the bars over the dimensionless variables $\bar{h}_r$, $\bar{h}_a$, $\bar{V}$, $\bar{\Delta}$, $\bar{y}$, and $\bar{W}$ are omitted for convenience:

$$d^2h_r/dy^2+\mathrm{Re}_m(1-V)h_a\Delta-h_r=0;\ d^2h_a/dy^2-\mathrm{Re}_m(1-V)(1+h_r)\Delta-h_a=0;\ V^2+\mathrm{Ha}^2h_a/\mathrm{Re}_m=0; \tag{8}$$

$$\frac{d\Delta}{dy} = -\frac{A}{V^2\Delta}\left[(1+h_r)\frac{dh_r}{dy}+h_a\frac{dh_a}{dy}\right]-\frac{1}{\mathrm{Fr}\,V^2};\ dW/dr=\Delta.$$

The boundary conditions for system (8) have the following form

$$\text{when } y=-a\pi/\tau \quad h_a=0;\quad h_r=0;\quad W=0;$$
$$\text{when } y=a\pi/\tau \quad h_a=0;\quad h_r=0;\quad W=W_{fl}/2R\delta\tau. \tag{9}$$

System (8) gives a solution for the rotation of the fluid in the field of gravity. To obtain a solution for the problem under conditions of weightlessness, it is sufficient to set $\mathrm{Fr}=\infty$ in (8). It can be seen that also under weightless conditions the free surface of the fluid differs from the cylindrical surface due to f_y.

We now obtain an expression for losses in the liquid metal. Considering the nongradient flow and the variable slip s along y, we can express the total losses P_{lm} in the fluid in terms of friction losses P_{fr}:

$$dP_{lm} = dP_{fr}/(1-s).$$

We have for the differential dP_{fr}

$$dP_{fr} = (\lambda\varrho R V_{fi}^3\tau V^3/4)\,dy.$$

We now obtain the desired expression:

$$P_{lm} = (\lambda\varrho R V_{fi}^3\tau/4)\int_{-a\pi/\tau}^{a\pi/\tau} V^2 dy. \tag{10}$$

2. Solution of the System of Equations

Investigation of system (8) on a computer for the case of weightlessness indicated that for $\alpha \geq \tau$ the shape of the free surface differs very little from the cylindrical shape. Therefore, in the fourth equation in (8) the first term to the right of the equal sign can be neglected. Moreover, the characteristic values of s for the operation of the devices described in [1] are close to unity. In this regard, the magnitude of V by comparison to unity can be neglected in the first two equations in (8). Allowing for the above, one can obtain from (8):

$$d^2h_r/dy^2+\mathrm{Re}_m h_a dW/dy-h_r=0;$$
$$d^2h_a/dy^2-\mathrm{Re}_m(1+h_r)dW/dy-h_a=0; \tag{11}$$
$$d^2W/dy^2=K/h_a \quad (K=\mathrm{Re}_m/\mathrm{Fr}\cdot\mathrm{Ha}^2).$$

The solution of (11) is sought in the form of power series:

$$h_r=h_{r0}+Kh_{r1}+K^2h_{r2}+\ldots,$$
$$h_a=h_{a0}+Kh_{a1}+K^2h_{a2}+\ldots,$$
$$W=W_0+KW_1+K^2W_2+\ldots.$$

The parameter K for the devices described in [1] ranges from 0.01 to 0.06, which provides a rapid convergence of the power series. Substituting the power series into (11) and equating terms with the same powers of K, we obtain to the second-order approximation:

$$d^2h_{r0}/dy^2+\mathrm{Re}_m h_{a0}dW_0/dy-h_{r0}=0,$$
$$d^2h_{a0}/dy^2-\mathrm{Re}_m(1+h_{r0})dW_0/dy-h_{a0}=0, \tag{12}$$
$$h_{a0}d^2W_0/dy^2=0;$$

$$d^2h_{r1}/dy^2+\mathrm{Re}_m h_{a0}dW_1/dy+\mathrm{Re}_m h_{a1}dW_0/dy-h_{r1}=0,$$
$$d^2h_{a1}/dy^2-\mathrm{Re}_m(1+h_{r0})dW_1/dy-\mathrm{Re}_m h_{r1}dW_0/dy-h_{a1}=0, \tag{13}$$
$$h_{a1}d^2W_0/dy^2+h_{a0}d^2W_1/dy^2=1.$$

From the third equation in (12) with the condition (9), we obtain

$$W_0=\Delta_0\cdot y+w_{fl}/2, \tag{14}$$

where $w_{fl}=W_{fl}/2R\delta\tau$ is the dimensionless volume of the fluid and $\Delta_0=w_{fl}\tau/2\alpha\pi$ is the dimensionless thickness of the layer when the fluid surface is cylindrical.

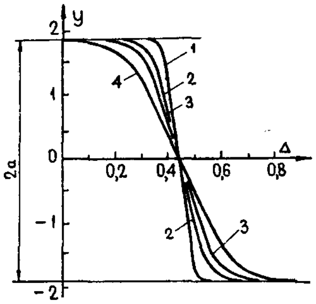

Fig. 2

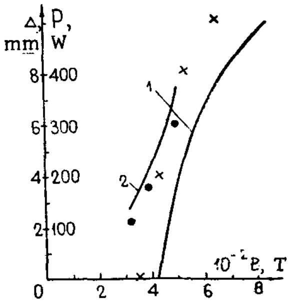

Fig. 3

Using (14), we obtain the following system instead of the 1st and 2nd equations in (12):

$$\frac{d^2h_{r0}}{dy^2}+\mathrm{Re}_m\Delta_0\cdot h_{a0}-h_{r0}=0;$$
$$\frac{d^2h_{a0}}{dy^2}-\mathrm{Re}_m(1+h_{r0})\Delta_0-h_{a0}=0. \qquad (15)$$

System (15) is in total agreement with the induction equation in [6]. Introducing the complex quantity $\dot{h}_0=h_{r0}+ih_{a0}$ we obtain instead of (15):

$$\frac{d^2\dot{h}_0}{dy^2}-t^2\dot{h}_0=i\,\mathrm{Re}_m\,\Delta_0;\quad t^2=1+i\,\mathrm{Re}_m\,\Delta_0.$$

Adapting the solution of [6] for this equation and allowing for (9), we obtain

$$\dot{h}_0=-\frac{i\,\mathrm{Re}_m\,\Delta_0}{t^2}\left[1-\frac{\mathrm{ch}\,ty}{\mathrm{ch}\,ta\pi/\tau}\right].$$

Separating out the imaginary part of $\dot{h}_0$, we obtain

$$h_{a0}=-L_1+L_2\cos L_3y\cdot\mathrm{ch}\,L_4y+L_5\sin L_3y\cdot\mathrm{sh}\,L_4y, \qquad (16)$$

where

$$L_1=\mathrm{Re}_m\Delta_0/(1+\mathrm{Re}_m^2\Delta_0^2);\quad L_2=2(L_1L_7-L_6L_8)/(L_7^2+L_8^2);$$
$$L_3=\sqrt{(\sqrt{1+\mathrm{Re}_m^2\Delta_0^2}-1)/2};\quad L_4=\sqrt{(1+\sqrt{1+\mathrm{Re}_m^2\Delta_0^2})/2};$$
$$L_5=2(L_1L_8+L_6L_7)/(L_7^2+L_8^2);\quad L_6=\mathrm{Re}_m^2\Delta_0^2/(1+\mathrm{Re}_m^2\Delta_0^2);$$
$$L_7=2\cos L_3a\pi/\tau\cdot\mathrm{ch}\,L_4a\pi/\tau;\quad L_8=2\sin L_3a\pi/\tau\cdot\mathrm{sh}\,L_4a\pi/\tau.$$

From the 3rd equation in (13), and using (9) and the evenness of the function h_{a0}, we obtain

$$W_1=\int\limits_{a\pi/\tau}^{y}\int\limits_{0}^{y}\frac{dy}{h_{a0}}dy. \qquad (17)$$

Considering the term W_1 in the expansion of W, we obtain via (14) and (17):

$$W=\frac{w_{\mathrm{fl}}}{2}+\Delta_0\cdot y+K\int\limits_{a\pi/\tau}^{y}\int\limits_{0}^{y}\frac{dy}{h_{a0}}\,dy.$$

Using this expression, the last equation in (8), and Eq. (16), we obtain the desired expression for the shape of the free surface:

$$\Delta=\Delta_0+K\int\limits_0^y\frac{dy}{-L_1+L_2\cos L_3y\cdot\mathrm{ch}\,L_4y+L_5\sin L_3y\cdot\mathrm{sh}\,L_4y}. \qquad (18)$$

Considering the term h_{a0} in the expansion of h_a and using the third equation in (8), we obtain the desired equation for the velocity distribution:

$$V^2=-(\mathrm{Ha}^2/\mathrm{Re}_m)[-L_1+L_2\cos L_3y\cdot\mathrm{ch}\,L_4y+L_5\sin L_3y\cdot\mathrm{sh}\,L_4y]. \qquad (19)$$

Substituting (19) into (10) and integrating, we obtain the expression for the losses in the liquid metal:

$$P_{\mathrm{lm}}=-(\lambda_0 R V_{\mathrm{fl}}^3\tau\,\mathrm{Ha}^2/4\mathrm{Re}_m)\{-L_12a\pi/\tau+$$
$$+[2L_2/(L_4^2+L_3^2)]\,[L_3\sin L_3a\pi/\tau\cdot\mathrm{ch}\,L_4a\pi/\tau+L_4\cos L_3a\pi/\tau\cdot\mathrm{sh}\,L_4a\pi/\tau]\}. \qquad (20)$$

3. Comparison with Experimental Data

Expressions (18) and (20) were calculated for the device described in [1] with different values of the induction B of the stator. The integral in (18) was determined numerically. Figure 2 shows the shape of the free surface of the fluid for $\mathrm{Re}_m=5.4$ with the following values of K:

$K=0.017$ (curve 1), $K=0.03$ (curve 2),
$K=0.04$ (curve 3), $K=0.06$ (curve 4).

In Fig. 3, curve 1 represents the width of the fluid ring at the upper nonconducting ring 4 (Fig. 1) as a function of the induction B. The crosses denote experimental data. The shift of the calculated curve to the right can be explained as due to the omission of capillary phenomena near the nonconducting ring 4, which should increase the width of the fluid ring resulting from the wetting factor.

In Fig. 3, curve 2 represents the losses in the liquid metal as a function of B. The points denote experimental data.

The width of the liquid-metal ring (eutectic indium–gallium–tin) was measured visually through a transparent Plexiglas nonconducting wall for a steady rotation of the metal. The losses in the liquid metal were determined experimentally by dividing the losses, which is similar to the procedure used for asynchronous electric motors. Considering the unavoidable measurement error and the approximate nature of the calculation, the agreement between them can be assumed to be satisfactory.

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