NUMERICAL ANALYSIS OF THREE-DIMENSIONAL MHD FLOW PROBLEMS

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UDC 538.4

The following three-dimensional problems are analyzed numerically: 1) flow past a cylinder of rectangular cross section and finite length; 2) flow over an inhomogeneity in the electrical conductivity of duct walls; 3) flow over a magnetic field inhomogeneity.

The number of papers devoted to the numerical analysis of three-dimensional problems in viscous fluid dynamics is extremely limited [1, 2]. Investigations of this nature are virtually nonexistent in magnetohydrodynamics [3]. In the latter realm, however, the investigation of three-dimensional problems takes on a special significance insofar as even problems that are well described in the absence of a magnetic field by two-dimensional models become intrinsically three-dimensional when a field is applied [4].

In the present article we investigate certain steady three-dimensional MHD flow problems in the noninductive approximation. The solvability of these problems has been studied earlier [5-7].

The steady-state solution is found by the relaxation method in the limit as $t \to \infty$ for the solution of the set of equations [3, 8]

$$\mathbf{v}_t - \mathbf{v}\times\mathbf{Z} = -\nabla\Pi + \mathrm{Re}^{-1}\nabla^2\mathbf{v} + \mathrm{N}\,\mathbf{j}\times\mathbf{B}; \tag{1_1}$$

$$\Pi_t + \nabla\mathbf{v} = 0; \quad \nabla^2\varphi = \mathbf{B}\cdot\mathbf{Z}, \qquad (1_2),\ (1_3)$$

in which $\Pi = p + \frac{1}{2}v^2$ is the total pressure of the fluid; $\mathbf{j} = \nabla\varphi + \mathbf{v}\times\mathbf{B}$ is the electric current density vector; $\mathbf{Z} = \nabla\times\mathbf{v}$ is the velocity vorticity vector; $\mathbf{B}$ is the external magnetic field vector ($\nabla\mathbf{B} = 0$; $\nabla\times\mathbf{B} = 0$); $\mathbf{v}$, p are the velocity vector and pressure of the fluid; φ is the potential, $\mathbf{E} = -\nabla\varphi$; and Re and N are the Reynolds and Stuart numbers.

The initial and boundary conditions are given below for each problem.

The set of equations (1) is approximated by finite-difference equations on two separate interlaced three-dimensional grids [2]. The first spatial derivatives in this case are replaced by central differences, and the first time (t) derivatives are replaced by forward differences. The quantities v and φ are determined on the main grid system from Eqs. (1_1) and (1_3), while Π is determined from Eq. (1_2) on the intermediate grid system. Here $\nabla\Pi$ in Eq. (1_1) is approximated on the main grid by means of the average values of Π on the intermediate grid with an order of approximation $O(h^2)$ (h is the spatial step of the grid). Accordingly, the quantity $\nabla_{\mathbf{v}}$ in Eq. (1_2) is approximated on the intermediate grid by means of the average values of v on the main grid, also with approximation order $O(h^2)$. Since the finite-difference analogs of Eqs. (1) are quite cumbersome, we give as an example only the finite-difference approximation of Eq. (1_2) on the intermediate grid system:

$$\begin{aligned}\Pi = \Pi_* - (\tau/4h)\,(&v'_{101} + v'_{111} + v'_{110} + v'_{100} - v'_{001} - v'_{011} - v'_{010} - v'_{000} + \\ &+ v_{010}{}^2 + v_{011}{}^2 + v_{110}{}^2 + v_{111}{}^2 - v_{000}{}^2 - v_{001}{}^2 - v_{100}{}^2 - v_{101}{}^2 + v_{001}{}^2 + \\ &+ v_{101}{}^3 + v_{011}{}^3 + v_{111}{}^3 - v_{000}{}^3 - v_{100}{}^3 - v_{010}{}^3 - v_{110}{}^3);\\ v^k{}_{i1,i2,i3} = v_k(x_1 + j_1 h;\ &x_2 + j_2 h;\ x_3 + j_3 h,\ t);\ 2j_k = i_k(5 - 3i_k);\ k = 1,\ 2,\ 3,\end{aligned} \tag{2}$$

where Π_* is the value of Π on the preceding time layer, τ is the time step, and $x = (x_1; x_2; x_3)$ are the Cartesian coordinates of the mesh points.

Translated from Magnitnaya Gidrodinamika, No. 2, pp. 35-40, April-June, 1973. Original article submitted September 28, 1972.

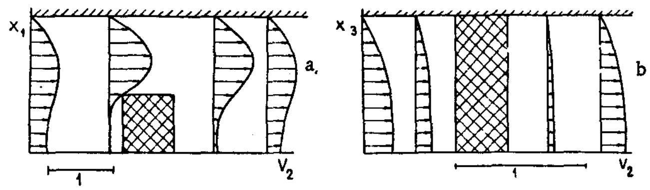

Fig. 1. Velocity profiles for flow past a rectangular cylinder of finite length in the absence of a magnetic field. a) Cross section $x_3 = 0.025$; b) $x_1 = 0.025$.

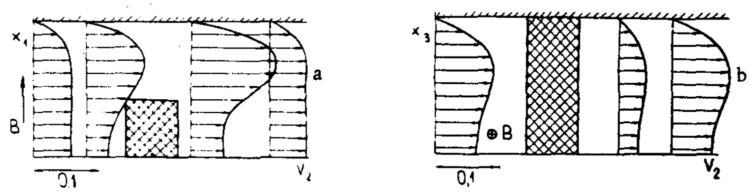

Fig. 2. Velocity profiles for flow past a rectangular cylinder in a transverse magnetic field. a) Cross section $x_3 = 0.025$; b) $x_1 = 0.025$.

The procedure for the solution of the set of difference equations is reduced to iterations of the Seidel type, the variables v and φ being more precisely determined in each iteration from the set of difference equations corresponding to (1_1) and (1_3) at each point of the grid simultaneously and then Π being calculated from (2), etc. One iteration consists of an explicit computation of all the difference equations from the origin in the direction of increasing values and back again. The calculations were carried out on a GE-415 computer.

1. Flow past a Rectangular Cylinder in a Rectangular Duct in a Crossflow Magnetic Field. We consider flow in a duct $|x| \leq (0.525;\ \infty;\ 0.525)$, $(|x_1| \leq 0.525;\ |x_2| \leq \infty;\ |x_3| \leq 0.525)$, in which are encountered periodically, with period 1, rectangular cylinders $|x_1| \leq 0.225$; $|x_3| \leq 0.525$. Taking the periodicity and symmetry of the problem into account, what we actually consider is a segment of the duct $(0;\ 0;\ 0) \leq x \leq (0.525;\ 1;\ 0.525)$ with an obstacle $(0;\ 0.4;\ 0) \leq x \leq (0.225;\ 0.6;\ 0.525)$. The boundary conditions are as follows: The flow is periodic in the cross sections $x_2 = 0$, 1; the corresponding variables are symmetrical and antisymmetrical in the planes $x_1 = 0$, $x_3 = 0$; at the rigid nonconducting walls the sticking condition holds ($v = 0$), and the normal component of the electric current is zero ($\partial\varphi/\partial n = 0$). Besides the conditions of periodicity, the pressure drop between the cross sections $x_2 = 0$, 1 is given:

$$\Delta p = p|_{x_2=0} - p|_{x_2=1}.$$

The initial conditions at $t = 0$ are as follows:

$$v \equiv 0; \quad \varphi \equiv 0; \quad \Pi = (1 - x_2)\Delta p;$$

v and φ are determined on the main grid $(11 \times 20 \times 11)$ with step $h = 0.5$:

$$x_1 = x_3 = (i - 0{,}5)h;\ i = 1, 2, \ldots, 11; \quad x_2 = ih;\ i = 1, 2, \ldots, 20; \tag{3_1}$$

and Π is determined on the intermediate grid $(11 \times 21 \times 11)$ with the same step:

$$x_1 = x_3 = (i - 1)h;\ i = 1, 2, \ldots, 11; \quad x_2 = (i - 0{,}5)h;\ i = 1, 2, \ldots, 21. \tag{3_2}$$

Thus, a nonlinear set of difference equations in 10,420 unknowns is solved. We first analyze the ordinary (nonmagnetic) hydrodynamical version.

For $\mathrm{Re} = 40$, $N = 0$, $\tau = 0.1$, and $\Delta p = 1$ the iteration procedure turns out to be divergent, as is consistent with the conditions obtained by the linear theory of the stability of difference schemes [9]:

$$\tau \leqslant \frac{8}{3\mathrm{Re}} + \left(\frac{64}{9\mathrm{Re}^2} + 4h^2 - 16h^2 v_0^2\right)^{1/2} \text{ for } v_0^2 < \frac{1}{4} + \left(\frac{2}{3\mathrm{Re}\,h}\right)^2, \tag{4_1}$$

$$2\tau \leqslant h\left(v_0 - \frac{2}{3\mathrm{Re}\,h}\right)^{-1} \text{ for } v_0 > \frac{2}{3\mathrm{Re}\,h}, \quad (v_0 = \max|v_i|). \tag{4_2}$$

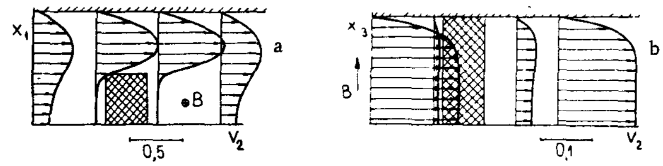

Fig. 3. Velocity profiles for flow past a rectangular cylinder in a magnetic field parallel to the long side of the cylinder. a) Cross section $x_3 = 0.025$; b) $x_1 = 0.025$.

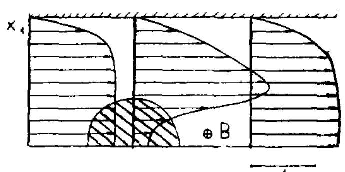

Fig. 4. Velocity profiles for flow over a conductivity inhomogeneity on a duct wall; cross section $x_3 = 0.05$.

For $\tau = 0.05$ the iteration procedure converges, and after 50 iterations (machine time ~ 30 min) we obtain a double-humped $v_2 - x_1$ profile (Fig. 1a) and an almost-parabolic $v_2 - x_3$ profile (Fig. 1b), the pressure p varying only slightly with x_3 but rather strongly with x_1, particularly in the neighborhood of the cylinder.

At the nodes $x_2 = 0.4$; $x_1 = 0.225$; $0 \leq x_3 \leq 0.525$ the pressure increases rapidly, and at the points $x_2 = 0.6$; $x_1 = 0.225$; $0 \leq x_3 \leq 0.525$ it decreases. In the plane $x_1 = 0.025$ the pressure is constant behind the body. The velocity v_2 becomes negative behind the cylinder [min $v_2 = -0.002$ at $x = (0.075;\ 0.65;\ 0.475)$], i.e., a stagnant flow zone is formed. In order to ascertain the accuracy with which the equation $\nabla \mathbf{v} = 0$ is satisfied, we point out that the mass flow of fluid across the sections $x_2 = \text{const}$ varies from 0.242 to 0.244, i.e., $Q = 0.243$.

In the two-dimensional case ($-\infty \leq x_3 \leq +\infty$) Q is approximately twice as large. Here $v_1 > 0$ for $x_2 < 0.5$, and $v_1 > 0$ for $x_2 > 0.5$; max $v_2 = 0.731$ for $x = (0.375;\ 0.5;\ 0.025)$.

We analyze two orientations of the magnetic field relative to the cylinder axis.

a) Let the magnetic field be perpendicular to the cylinder axis, i.e., $\mathbf{B} = (1;\ 0;\ 0)$. Then for parameters $\mathrm{Re} = 40$, $N = 10$, $\tau = 0.05$, and $\Delta p = 1$ it turns out that the component v_3 is small, Q is smaller than in the case $N = 0$ ($Q = 0.07$), the $v_2 - x_1$ profile approaches a Hartmann profile (Fig. 2a), the $v_2 - x_3$ profile becomes double-humped (Fig. 2b), and $v_2 \geq 0$, where max $v_2 = 0.190$ for $x = (0.375;\ 0.5;\ 0.325)$.

With an increase in the parameter N ($N = 100$, $\Delta p = 3$) the flow becomes symmetrical about the plane $x_2 = 0.5$, i.e., the components v_1, v_3, j_2 and v_2, j_1, j_3 become antisymmetrical and symmetrical, respectively, about the plane $x_2 = 0.5$. This result implies that the flow does not differ much from the Stokes regime. The component v_3 is strongly suppressed by the magnetic field; the components v_1 and v_2 acquire the same order of magnitude; the $v_2 - x_3$ profile is clearly double-humped with a smooth middle part; $Q = 0.3$.

b) Let the magnetic field be parallel to the cylinder axis, i.e., $\mathbf{B} = (0,\ 0,\ 1)$. For parameters $\mathrm{Re} = 40$, $N = 10$, $\tau = 0.05$, and $\Delta p = 3$ the following results are now obtained: The component v_3 is suppressed by the magnetic field, Q approximately coincides with the flow rate for $N = 0$ ($Q = 0.25$), the $v_2 - x_1$ profile becomes double-humped (Fig. 3a), the $v_2 - x_3$ profile approaches a Hartmann profile (Fig. 3b), $v_2 \geq 0$, i.e., vortices are not observed, and the electrical potential φ is almost constant with respect to x_3, i.e., $E_3 \approx 0$.

A comparison of Fig. 1b and Fig. 3b discloses a distinct tendency of the flow to go over to two-dimensional flow in a plane perpendicular to the magnetic field. We note that the constant Δp (= 3) is chosen in this case so as to make the flow rate the same as without the field.

2. Flow over an Electrical Conductivity Inhomogeneity of Duct Walls in a Magnetic Field Transverse to the Inhomogeneous Walls. We now consider flow in a duct $|x| \leq (1.95;\ \infty;\ 0,\ 35)$, supposing that its nonconducting walls $x_3 = \pm 0.35$ [which are perpendicular to the field $\mathbf{B} = (0,\ 0,\ 1)$] include circular zones $x_1^2 + (x_2 - 2)^2 \leq 0.49$ possessing infinite conductivity ($\varphi = 0$).

We actually consider the domain $(0;\ 0;\ 0) \leq x \leq (1.95;\ 6;\ 0.35)$ with symmetry conditions in the planes $x_1 = 0$, $x_3 = 0$. A Shercliff profile, i.e., the steady-state solution of the initial system (1) in a smooth duct with nonconducting walls for a given fluid flow rate $Q = 2.73$ and for $v_1 = v_3 \equiv 0$, is specified at the duct inlet $x_2 = 0$:

$$v_2 = \frac{2\mathrm{Re}\,\Delta p}{a} \sum_{i=0}^{\infty} (-1)^i \lambda_i^{-3} \cos \lambda_i x_1 \left(1 + \mathrm{sh}^{-1}(k_1 - k_2) b\right) (\mathrm{ch}\, k_1 x_3\, \mathrm{sh}\, k_2 b - \mathrm{ch}\, k_2 x_3\, \mathrm{sh}\, k_1 b); \qquad (5_1)$$

$$\varphi=\frac{2\mathrm{Re}\,\Delta p}{a}\sum_{i=0}^{\infty}(-1)^i\lambda_i^{-3}\sin\lambda_i x_1(\lambda_i^{-1}+\lambda_i\,\mathrm{Ha}^{-1}\,\mathrm{sh}^{-1}(k_1-k_2)b)(k_1^{-1}\,\mathrm{ch}\,k_1x_3\,\mathrm{sh}\,k_2b-k_2^{-1}\,\mathrm{ch}\,k_2x_3\,\mathrm{sh}\,k_1b), \tag{5$_2$}$$

$$\frac{\Delta p\,\mathrm{Re}}{a}=\frac{Q}{8}\left(\sum_{i=0}^{\infty}\lambda_i^{-4}(b+\lambda_i^{-2}(k_1-k_2)\,\mathrm{sh}^{-1}(k_1-k_2)b\,\mathrm{sh}\,k_1b\,\mathrm{sh}\,k_2b\right)^{-1};$$

$$k_{1,2}=-\mathrm{Ha}/2\pm(\mathrm{Ha}^2/4+\lambda_i^2)^{1/2};\ \lambda_i=\frac{2i+1}{2a}\pi;\ a=1.95;\quad b=0.35.$$

Weak conditions are imposed on the duct exit $x_2=6$ in the form

$$\frac{\partial v}{\partial x_2}=\frac{\partial\varphi}{\partial x_2}=0. \tag{6}$$

The initial conditions at $t=0$ include the values of $\mathbf{v}$, φ given by (5) and

$$\Pi=(6-x_2)\Delta p;\quad \Delta p=2.723; \tag{7}$$

$\mathbf{v}$, φ are determined on the main grid system $(20\times30\times4)$ with step $h=0.1$:

$$x_1=(i-0.5)h;\ i=1,2,\ldots;\ 20;\quad x_3=(i-0.5)h;\ i=1,2,3,4;\ x_2=2ih;\ i=1,2,\ldots;\ 30,$$

and Π is determined on the intermediate grid $(20\times30\times4)$ with the same step:

$$x_1=(i-1)h;\ i=1,2,\ldots;\ 20;\quad x_3=(i-1)h;\ i=1,2,3,4;\ x_2=(i-0.5)2h,\ i=1,2,\ldots;\ 30,$$

i.e., a set of difference equations in 9012 unknowns is solved.

The calculations for $\mathrm{Re}=20$, $N=10$, and $\tau=0.03$ show that the flow is greatly inhibited in the vicinity of the conductivity inhomogeneity (Fig. 4) and the v_2-x_1 profile acquires a double-humped structure. This result is consistent with the experimental results of [10]. The v_2-x_3 profile thickens and acquires a Hartmann shape.

We have also undertaken flow calculations for the case in which the conducting zones are rectangular, coinciding with the base of the cylinder analyzed in the preceding sections, and are spaced periodically with the same period. For $\mathrm{Re}=40$, $N=10$, and $\Delta p=3$ the v_2-x_1 profile is double-humped and varies only slightly with x_2, the same being true with respect to x_3 except in the immediate proximity of the walls $x_3=\mathrm{const}$. The variables v_1, v_3, j_2, and j_3 are small in comparison with v_2 and j_1; $v_2\geq0$, $j_1>0$, and $Q=0.37$.

3. Flow over a Magnetic Field Inhomogeneity in a Duct with Nonconducting Walls. The configuration of the analytical region is the same as in problem 2. The inhomogeneous magnetic field is created by means of two semiinfinite cylindrical solenoids attached to the two parallel duct walls $x_3=+0.35$, with axis at the point $x=(0,2,0)$ and with radius 0.7. The magnetic field components B_r and B_z are shown in cylindrical coordinates $z=x_3$, $r^2=x_1^2+(x_2-2)^2$ in Fig. 5. The values of B_1 and B_2 are obtained from the relations $B_1=\frac{x_1}{r}B_r$, $B_2=\frac{x_2-2}{r}B_r$ with linear interpolation. The average value is $B_z^2\approx12$. In the duct cross sections $x_2=0$, $x_2=6$ it is given that $\varphi=0$. Conditions (6) are applied to the vector $\mathbf{v}$ at the exit $x_2=6$, and the stationary solution of the system (1) for $N=0$, i.e., a Boussinesq profile, is specified at the entry:

$$v_2=\Delta p\,\mathrm{Re}\left(\frac{a^2-x_1^2}{2}-\frac{16a^2}{\pi^3}\sum_{k=0}^{\infty}(-1)^k(2k+1)^{-3}\,\mathrm{ch}^{-1}\frac{(2k+1)\pi b}{2a}\,\frac{\mathrm{ch}(2k+1)\pi x_3}{2a}\cos\frac{(2k+1)\pi x_1}{2a}\right);$$

$$\Delta p\,\mathrm{Re}=\left(\frac{4}{3}a^3b-\frac{256a^4}{\pi^5}\sum_{k=0}^{\infty}(2k+1)^{-5}\,\mathrm{th}\frac{(2k+1)\pi b}{2a}\right)^{-1}Q=27.62; \tag{8}$$

$$a=1.95;\quad b=0.35;\quad Q=2.73.$$

For the initial conditions at $t=0$ we specify the values of v from (8), $\varphi\equiv0$, and the values of Π from (7), where $\Delta p=1.381$. A numerical calculation for $\mathrm{Re}=20$, $N=1$, and $\tau=0.03$ shows that the v_2-x_1 profile has a double-humped shape except in the immediate proximities of the entry and exit.

The pressure drop Δp increases by about 6% over the value for $N=0$, and $j_3<0$ only in the vicinity of the solenoid. If the Stuart number is increased ($N=3$) a closed-circulation region is formed in the magnetic field inhomogeneity zone, the rest of the flow moving around that region (Fig. 6a). The pressure drop Δp increases by 9% over the case of $N=0$:

$$\min v_2=-0.074$$

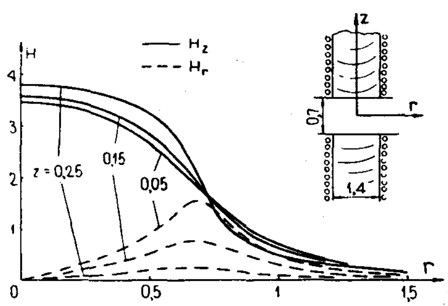

Fig. 5

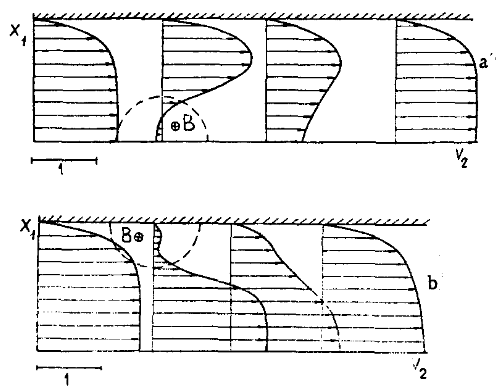

Fig. 6

Fig. 5. Distribution of the components of a magnetic field created by two cylindrical semiinfinite solenoids.

Fig. 6. Velocity profiles for flow over a magnetic field inhomogeneity. a) Symmetrical flow; b) asymmetrical flow; cross section $x_3 = 0.05$.

for x = (0.15; 2.0; 0.05);

$$\max v_2 = 1.711$$

for x = (1.35; 2.2; 0.05).

We conclude with an analysis of a variation on the preceding problem wherein the axis of the half-solenoid is situated at the point x = (1.95; 2; 0), i.e., at the duct wall $x_1 = 1.95$. The results of the corresponding numerical calculation are shown in Fig. 6b for Re = 20, N = 10, and $\tau = 0.03$.

As in the preceding case, the fluid flows over the region with the magnetic field. This situation has another specific characteristic: In the region with the magnetic field another velocity maximum is formed. The reasons for the formation of this maximum are well known (see, e.g., [11]) and are related to the inception of vorticity due to the inhomogeneity of the magnetic field near the nonconducting wall.

We note in conclusion that for problems involving periodic conditions the value that we have given for the Reynolds number does not correspond to the mass flow rate, because for such problems it is necessary to specify the pressure drop and to compute the flow rate. It is necessary to multiply the given values by 0.91Q in order to obtain the Reynolds number for the flow rate.

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