# SOLUTION OF A MAGNETOHYDRODYNAMIC BOUNDARY-VALUE PROBLEM WITH A PERIODIC BOUNDARY CONDITION

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The steady flow of a viscous incompressible isotropically conducting fluid is considered in a plane channel with the conductivity of the walls being an arbitrary periodic function. The solution of the problem is in the form of an expansion in trigonometric series with coefficients satisfying an infinite linear algebraic system. Justification is provided for reducing the problem to the solution of this system. The results are given of calculations for a channel with walls having periodically alternating sections with different conductivity. The effect of the conductivity of the different sections of the channel walls on the velocity of the fluid u is ascertained, as is the dependence of $\delta = u_{max} - u_{min}$ on the periodicity parameter.

## 1. Formulation of the Problem.

We consider the steady flow of a viscous incompressible isotropically conducting fluid in a plane-parallel channel in the presence of a transverse homogeneous magnetic field $B_0$. The fluid flows along the z axis under a constant pressure gradient $\partial p/\partial z = -P = \text{const}$.

The solution of this problem has been obtained in [1-4] when the walls are either nonconducting or have an arbitrary constant conductivity.

Below, a solution is obtained for the problem of the flow in a channel with the conductivity of the walls being an arbitrary periodic function $\sigma_W(x) \geq 0$. In this case we can restrict ourselves to finding a solution in the region $G\{-l \leq x \leq l, -1 \leq y \leq 1\}$, where $l = a/L$ is a dimensionless parameter characterizing the ratio of the width of the channel to the length of the period. In this case one should assume $\sigma_W(x) = \sigma_W(x + 2ln)$, $n = 0, \pm 1, \pm 2, \ldots$

The magnetohydrodynamic equations allow us to reduce the solution of the problem to finding two functions $u_z(x, y) = PL^2/\eta u(x, y)$ and $B_z(x, y) = \mu_0 P(\sigma/\eta)^{1/2} L^2 B(x, y)$ corresponding to the distributions in the channel of the axial fluid velocity and the induced magnetic field from the following system of equations:

$$\Delta u + \mathrm{Ha}\frac{\partial B}{\partial y} = -1; \quad \Delta B + \mathrm{Ha}\frac{\partial u}{\partial y} = 0;$$
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; \quad \mathrm{Ha} = B_0 L\left(\frac{\sigma}{\eta}\right)^{1/2}. \tag{1}$$

Here, Ha is the Hartmann number, $\sigma$ is the electrical conductivity of the fluid, $\eta$ is the dynamic viscosity, and $\mu_0$ is a magnetic constant.

We seek solutions of (1) that are periodic in x and that for $y = \pm 1$ satisfy the boundary conditions [5]

$$u = 0, \quad \pm\frac{\partial B}{\partial y} + \Theta(x)\cdot B = 0, \quad \Theta(x) = \frac{\sigma}{\sigma_w(x)}\frac{L}{d}, \tag{2}$$

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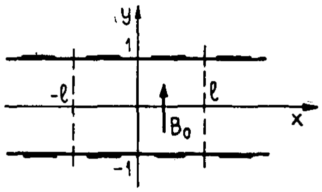

Fig. 1

where the parameter Θ(x) characterizes the conductivity of thin channel walls with thickness d and is a periodic function of the x coordinate. Furthermore, we assume that Θ(x) is even in the interval $-l \le x \le l$.

## 2. Construction of the Solution.

We consider the function [1]

$$V(y) = \frac{1}{\mathrm{Ha}} \frac{\mathrm{ch\,Ha} - \mathrm{ch\,Ha}y}{\mathrm{sh\,Ha}}, \quad H(y) = \frac{1}{\mathrm{Ha}} \left( \frac{\mathrm{sh\,Ha}\,y}{\mathrm{sh\,Ha}} - y \right). \tag{3}$$

With the substitution

$$u(x, y) = V(y) + v(x, y);$$
$$B(x, y) = H(y) + h(x, y)$$

the boundary-value problem (1), (2) reduces to the system of equations

$$\Delta v + \mathrm{Ha} \frac{\partial h}{\partial y} = 0, \quad \Delta h + \mathrm{Ha} \frac{\partial v}{\partial y} = 0 \tag{4}$$

with respect to the functions v(x, y) and h(x, y), which are even in x and which satisfy the boundary conditions

$$v = 0 \quad \text{for} \quad y = \pm 1; \tag{5}$$

$$\pm \frac{\partial h}{\partial y} + \Theta(x) h = \mp q \quad \text{for} \quad y = \pm 1;$$
$$q = \mathrm{cth\,Ha} - \mathrm{Ha}^{-1} \tag{6}$$

and the periodicity conditions.

The solution of (4), which is even and periodic in x, is

$$v(x, y) = \sum_{k=0}^{\infty} \left( a_k \frac{\mathrm{ch}\,\alpha_k y}{\mathrm{ch}\,\alpha_k} + b_k \frac{\mathrm{ch}\,\beta_k y}{\mathrm{ch}\,\beta_k} \right) \cos \frac{k\pi x}{l},$$
$$h(x, y) = -\sum_{k=0}^{\infty} \left( a_k \frac{\mathrm{sh}\,\alpha_k y}{\mathrm{ch}\,\alpha_k} + b_k \frac{\mathrm{sh}\,\beta_k y}{\mathrm{ch}\,\beta_k} \right) \cos \frac{k\pi x}{l} \tag{7}$$
$$\alpha_k, \beta_k = {}^1\!/_2 (\mathrm{Ha} \pm \omega_k), \quad \omega_k = \left( \mathrm{Ha}^2 + \frac{4k^2\pi^2}{l^2} \right)^{1/2}.$$

The coefficients $a_k$ and $b_k$ in the expansions (7) should be determined from the boundary conditions (5) and (6). Condition (5) for v(x, y) from (7) leads to the following relation between $a_k$ and $b_k$

$$a_k + b_k = 0. \tag{8}$$

Then h(y, x) from (7), with (8) taken into account, can be represented by

$$h(x, y) = -\sum_{k=0}^{\infty} a_k \left( \frac{\mathrm{sh}\,\alpha_k y}{\mathrm{ch}\,\alpha_k} - \frac{\mathrm{sh}\,\beta_k y}{\mathrm{ch}\,\beta_k} \right) \cos \frac{k\pi x}{l}$$

which, upon substitution into boundary condition (6), leads to

$$\sum_{k=0}^{\infty} a_k \omega_k \cos \frac{k\pi x}{l} + \Theta(x) \sum_{k=0}^{\infty} a_k \eta_k \cos \frac{k\pi x}{l} = q,$$
$$\eta_k = \mathrm{th}\,\alpha_k - \mathrm{th}\,\beta_k. \tag{9}$$

We now demonstrate a method for finding the sequence $a_k$ from (9). We represent the even function Θ(x) by an expansion in a Fourier series in the interval $-l \le x \le l$

$$\Theta(x) = \frac{\vartheta_0}{2} + \sum_{k=1}^{\infty} \vartheta_k \cos \frac{k\pi x}{l} \tag{10}$$

and we introduce an auxiliary function

$$\Psi(x)=\Theta(x)\sum_{k=0}^{\infty} a_k\eta_k \cos\frac{k\pi x}{l}. \tag{11}$$

Since $\Psi(x)$ is an even function, its expansion in a Fourier series in the interval $-l\leq x\leq l$ can be represented by

$$\Psi(x)=\frac{\lambda_0}{2}+\sum_{k=1}^{\infty}\lambda_k \cos\frac{k\pi x}{l}. \tag{12}$$

On the basis of the multiplication rule for Fourier series [6], we find

$$\lambda_k=a_0\eta_0\vartheta_k+\frac{1}{2}\sum_{j=1}^{\infty}a_j\eta_j(\vartheta_{k-j}+\vartheta_{k+j}),\qquad k=0,1,2,\ldots, \tag{13}$$

where in (13) one should set $\vartheta_{-n}=\vartheta_n$.

Taking (10)-(13) into account, we now have from (9)

$$a_0\left(\omega_0+\frac{\vartheta_0}{2}\eta_0\right)+\frac{1}{2}\sum_{j=1}^{\infty}a_j\eta_j\vartheta_j=q; \tag{14}$$

$$a_k\omega_k+a_0\eta_0\vartheta_k+\frac{1}{2}\sum_{j=1}^{\infty}a_j\eta_j(\vartheta_{k-j}+\vartheta_{k+j})=0,\qquad k=1,2,\ldots$$

Noting that $\omega_0=\mathrm{Ha}$ and $\eta_0=\tanh\mathrm{Ha}$, we obtain from the first equality in (14)

$$a_0=\left(\mathrm{Ha}+\frac{\vartheta_0}{2}\,\mathrm{th\,Ha}\right)^{-1}\left(q-\frac{1}{2}\sum_{j=1}^{\infty}a_j\eta_j\vartheta_j\right). \tag{15}$$

Then eliminating $a_0$ from the second equality in (14), we arrive at an infinite system of linear algebraic equations with respect to the desired coefficients $a_k$

$$a_k+\sum_{j=1}^{\infty}C_{kj}a_j=\nu_k,\qquad k=1,2,\ldots,$$

$$\nu_k=-\frac{\vartheta_k}{\omega_k}\cdot\frac{q\,\mathrm{th\,Ha}}{\mathrm{Ha}+{}^1/_2\vartheta_0\,\mathrm{th\,Ha}},$$

$$C_{kj}=\frac{\eta_j}{2\omega_k}\left(\vartheta_{k-j}+\vartheta_{k+j}-\frac{\mathrm{th\,Ha}}{\mathrm{Ha}+{}^1/_2\vartheta_0\,\mathrm{th\,Ha}}\vartheta_k\vartheta_j\right). \tag{16}$$

We will show that for the matrix elements $C_{kj}$ and the free terms $\nu_k$ of the infinite system (16) there are estimates

$$\sum_{k,j=1}^{\infty}|C_{kj}|^2<\infty,\qquad \sum_{k=1}^{\infty}|\nu_k|^2<\infty, \tag{17}$$

i.e., to find the coefficients $a_k$ from (16) we employ the reduction method [7], taking as the approximate solution of the infinite system (16) the solution of the corresponding truncated system

$$a_k^*+\sum_{j=1}^{N}C_{kj}a_j^*=\nu_k,\qquad k=1,2,\ldots,N,$$

where the values of $a_k^*$ converge to $a_k$ as $N\to\infty$.

Actually, since $\Theta(x)\geq 0$ then $\vartheta_0\geq 0$ and $|\vartheta_k|\leq\vartheta_0$; therefore

$$|C_{kj}|^2\leqslant \mathrm{const}\,\frac{1}{\omega_k^2}\left(|\vartheta_{k-j}|^2+|\vartheta_{k+j}|^2+|\vartheta_j|^2\right),$$

but then

$$\sum_{k,j=1}^{\infty}|C_{kj}|^2\leqslant \text{const}\sum_{k=1}^{\infty}\frac{1}{\omega_k^2}\Big(\sum_{j=-\infty}^{k-1}\vartheta_j^2+\sum_{j=k+1}^{\infty}\vartheta_j^2+\sum_{j=1}^{\infty}\vartheta_j^2\Big)\leqslant \text{const}\sum_{k=1}^{\infty}\Big(\frac{1}{k^2}\sum_{j=0}^{\infty}\vartheta_j^2\Big). \tag{18}$$

Since the $\vartheta_j$ are Fourier coefficients, the series

$$\sum_{j=0}^{\infty}\vartheta_j^2$$

converges, and it then follows from (18) that the estimate (17) is valid for the matrix $C_{jk}$. The validity of (17) for $\nu_k$ is obvious.

Thus, the solution of the original problem (1), (2) can be represented in the form

$$\begin{aligned} u(x,y)&=\frac{1}{\text{Ha}}\frac{\text{ch Ha}-\text{ch Ha}\,y}{\text{sh Ha}}+\sum_{k=0}^{m}a_k\Big(\frac{\text{ch}\,\alpha_k y}{\text{ch}\,\alpha_k}-\frac{\text{ch}\,\beta_k y}{\text{ch}\,\beta_k}\Big)\cos\frac{k\pi x}{l};\\ B(x,y)&=\frac{1}{\text{Ha}}\Big(\frac{\text{sh Ha}\,y}{\text{sh Ha}}-y\Big)-\sum_{k=0}^{\infty}a_k\Big(\frac{\text{sh}\,\alpha_k y}{\text{ch}\,\alpha_k}-\frac{\text{sh}\,\beta_k y}{\text{ch}\,\beta_k}\Big)\cos\frac{k\pi x}{l}, \end{aligned} \tag{19}$$

where the coefficients $a_k$ satisfy the infinite system (16), the solution for which is obtained by the reduction method.

Note that this solution can be used to obtain the solutions of the following limiting problems [1-4].

1) Steady Flow in a Plane Channel with Ideally Conducting Walls. In this case $\Theta(x)\equiv 0$ and in the infinite system (16) all $C_{kj}=0$ and $\nu_k=0$. Consequently, system (16) has a solution $a_k=0$ for $k=1, 2, \ldots$, and we have from (15)

$$a_0=\frac{\text{cth Ha}}{\text{Ha}}-\frac{1}{\text{Ha}^2}.$$

2) Flow in a Plane Channel with Walls Having a Constant Finite Conductivity. For this problem $\Theta(x)=\Theta\equiv$ const, i.e., $\vartheta_0=2\Theta$ and all $\vartheta_k=0$ for $k=1, 2, \ldots$. In this case the matrix of (16) is diagonal, and since $\nu_k=0$ then $a_k=0$ for $k=1, 2, \ldots$, and we find from (15)

$$a_0=(\text{Ha}+\Theta\,\text{th Ha})^{-1}(\text{cth Ha}-\text{Ha}^{-1}).$$

3) Flow in a Channel with Nonconducting Walls. Executing in (16) a limiting transition $\vartheta_0\to\infty$ and $\vartheta_k\to 0$ for $k=1, 2, \ldots$, we find from (15) and (16) that $a_k=0$ for all $k=0, 1, 2, \ldots$, and the solution (19) of this limiting problem coincides with the known solution (3).

## 3. Results of the Calculations.

The formulas (19) can be used to calculate the axial velocity distribution of the fluid and the distribution of the induced magnetic field for specified functions $\Theta(x)$.

As an example, we consider the case when the function $\Theta(x)$ in the interval $-l\leq x\leq l$ has the form

$$\Theta(x)=\begin{cases}\Theta_1=\text{const} & \text{for } |x|<{}^1/_4 l,\ {}^3/_4 l<|x|<l,\\ \Theta_2=\text{const} & \text{for } {}^1/_4 l<|x|<{}^3/_4 l.\end{cases}$$

The Fourier coefficients of this function in (15) and (16) are given by

$$\vartheta_0=\Theta_1+\Theta_2,$$
$$\vartheta_k=(-1)^k\frac{4}{\pi k}(\Theta_2-\Theta_1)\sin\frac{k\pi}{4}.$$

Specifying $\Theta(x)$ in this manner corresponds to a channel (Fig. 1) whose walls consist of alternating sections with different conductivity (for explicitness, sections with the same conductivity are not shown in the figure).

Figure 2 gives the velocity distribution of the fluid in the center of the channel ($y=0$) for Ha = 30, $\Theta_1=10^3$, $\Theta_2=10^{-3}$ for different values of the period $l$. Figure 2a corresponds to $l=2$ and Fig. 2b to $l=0.5$.

As can be seen from the figures, the velocity of the fluid depends on the conductivity at the corresponding sections of the walls. Maximum velocity $u_{max}$ is obtained at the center of sections with smaller

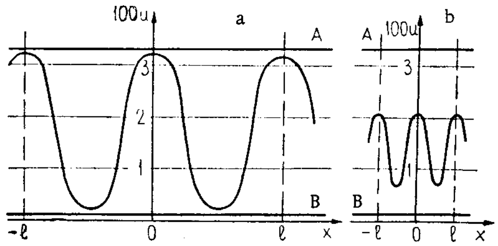


Fig. 2

wall conductivity, and the minimum velocity $u_{min}$ is at the center of sections with larger wall conductivity (the retardation effect [8, 9]).

The value of $u_{max}$ does not exceed the value of the velocity $u_1$ in a channel where the walls have constant $\Theta(x) = \Theta_1$ conductivity (line A in Fig. 2), which corresponds to the solution of the limiting problem 2. The value of $u_{min}$ is larger than the corresponding value of the velocity $u_2$ in the limiting problem 2 for $\Theta(x) = \Theta_2$ (line B in Fig. 2).

Comparison of the velocity profiles in Figs. 2a and 2b indicates that $l$ affects the magnitude of $\delta = u_{max} - u_{min}$. As the calculations imply, $\delta$ increases, approaching $\delta_{max} = u_1 - u_2$ as the period $l$ increases. As $l$ decreases $\delta \to 0$; this is obviously due to the viscous forces. The velocity distribution over y is a symmetric profile with a maximum at $y = 0$.

In conclusion we note that reducing the infinite system (16) assumes the existence of a solution of this system satisfying the condition

$$\sum_{k=1}^{\infty} a_k^2 < \infty.$$

If $\Theta(x)$ is continuous, then the existence and uniqueness of the solution of (16) follows from the regularity of this system [10].

Actually, since

$$\sum_{j=1}^{\infty} |C_{kj}| < \frac{2}{\omega_k} \sum_{j=0}^{\infty} |\vartheta_j|$$

and for a continuous function

$$\sum_{j=0}^{\infty} |\vartheta_j| < \infty,$$

then we have the condition for quasiregularity of (16)

$$\sum_{j=1}^{\infty} |C_{kj}| < \infty \quad \text{for } k = 1, 2, \ldots, n;$$

$$\sum_{j=1}^{\infty} |C_{kj}| < 1 \quad \text{for } k = n+1, n+2, \ldots$$

Furthermore, for large Ha the sum of the moduli of the matrix elements in each row can become smaller than $\varepsilon < 1$ and this means that (16) is completely regular.

This method of solving the problem can be extended to the case of a periodic dependence not involving an even function.

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