AUTO-OSCILLATIONS OF A CONDUCTING PISTON

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The article considers the motion of an electrically conducting piston in a sealed tube, filled with an ideal gas, having heated ends and cooled side walls. The piston is located in a homogeneous constant magnetic field, and the electrode parts of the side walls are short-circuited through the load. It demonstrates the existence of a threshold ratio of the temperatures of the hot and cold walls; as this ratio is exceeded, auto-oscillations arise, and there is a conversion of thermal energy into electrical. The frequency of the auto-oscillations and the efficiency of the device are calculated.

The monograph [1] pointed to the possibility of auto-oscillations of an electrically conducting liquid in a cooled channel whose ends are connected to two heated boilers. With the aim of a qualitative investigation of such a motion, let us examine the problem in the following idealized statement (Fig. 1). In the middle of a sealed tube there is an electrically conducting piston 3, separating the cavities 1 and 4, filled with an ideal gas. The piston is located in a homogeneous constant magnetic field, oriented along the z axis; the lateral electrode walls 2 are short-circuited through the load R. If the piston is not subjected to the action of braking forces and the walls are heat-insulated, then a displacement of the piston away from the center of the tube brings about undamped harmonic vibrations. The role of an elastic link is played by the gas in the cavities on both sides of the piston. If the walls admit of heat transfer, and their temperature is maintained constant, the vibrations will be damped. In actuality, a displacement of the piston, bringing about compression of the gas in one of the cavities, leads to an excess of the temperature over the temperature of the walls and, consequently, to heat transfer through the walls. Therefore, the temperature and pressure of the gas in this cavity will be less than in the adiabatic case. In the other cavity, the pressure will be higher than for heat-insulated walls. Thus, on the piston there will act a pressure difference which is less than in the adiabatic case, which leads to damping of the vibrations. If, now, the ends of the tube are heated to the temperature T_w and the side walls are cooled to the temperature T'_w, then starting from some threshold value of the temperature T_w the vibrations become undamped, even in the presence of braking forces. The temperature T_w must be sufficiently high so that with the displacement of the piston and its compression of the gas in one of the cavities there will be an additional increase in the temperature and, consequently, of the pressure in comparison with the case of adiabatic compression. The additional pressure difference arising compensates the action of the forces braking the piston, which leads to auto-oscillations. There is the usual closed circular process of the conversion of heat into work: the compressed gas is heated; the heated gas expands; the expanding gas is cooled; and the cooled gas is compressed. A shortcoming of such a cycle is the inevitable "outrunning" of the supply of heat and "outrunning" of the cooling. The supply of heat takes place over the time of the whole course of the compression, due to a decrease in the heat-transfer surface, and the cooling over the whole course of the expansion. In addition, the gas is simultaneously in communication with hot and cold sources of heat, which sets up a heat flux from the hot source to the cold source, although there is no thermal conductivity inside the gas. Nevertheless, in some cases, the work of the cycle is

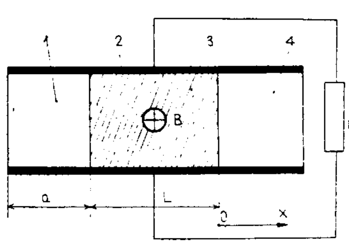

Fig. 1. Statement of problem.

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sufficient to maintain auto-oscillations, even with braking of the piston by an electromagnetic force, when it gives up energy to the external circuit.

In setting up the equations, the assumption is made that over each of the volumes of gas the temperature is distributed uniformly. It depends only on the time, and it is established under the action of heat transfer through the walls (with an identical heat-transfer coefficient) and on the change in the volume. The mass of the gas is assumed to be negligibly small in comparison with the mass of the piston, so that the momentum of the gas is neglected. The temperatures of the ends and side walls are maintained constant. It is postulated that the piston does not exchange heat with the gas; the friction of the piston at the wall is not taken into consideration. In the problem there arises a natural scale for the velocity v^*, from the equality of the kinetic energy of the piston and the work of the forces of pressure in the length a:

$$\varrho_p F L v^* \sim pFa.$$

The system of starting equations (energy, state, continuity, influx of heat to the gas, and the momentum of the piston) has the form

$$\frac{dQ_1}{dt}=c_v\frac{dT_1}{dt}+p_1\frac{d}{dt}\frac{1}{\varrho_1};\quad \frac{dQ_2}{dt}=c_v\frac{dT_2}{dt}+p_2\frac{d}{dt}\frac{1}{\varrho_2};\tag{1), (2}$$

$$p_1=\varrho_1 gRT_1;\quad p_2=\varrho_2 gRT_2;\tag{3}$$

$$\varrho_1=G/S_1(a-x);\quad \varrho_2=G/S_1(a+x);\tag{4}$$

$$\frac{dQ_1}{dt}=\frac{\alpha S_1}{G}(T_w-T_1)+\frac{\alpha S_2}{G}(a-x)(T'_w-T_1);\tag{5}$$

$$\frac{dQ_2}{dt}=\frac{\alpha S_1}{G}(T_w-T_2)+\frac{\alpha S_2}{G}(a+x)(T'_w-T_2);\tag{6}$$

$$\varrho_p\frac{dv}{dx}=\sigma B^2 a(\eta_e-1)-\frac{p_1-p_2}{L};\quad \frac{dx}{dt}=v.\tag{7), (8}$$

The following notation is adopted: G is the mass of the gas in the cavity; S_2 is the perimeter of the tube; S_1 is the area of the surface of an end; α is the heat-transfer coefficient; Q_1 and Q_2 are the amounts of heat arriving through the walls; η_e is the electrical efficiency; the subscripts 1 and 2 relate to the parameters of the gas in the different cavities; the subscript p relates to the piston; the remaining notation is the usual.

For a transition of the equations to dimensionless form, the following quantities are selected as basic:

$$l^*=a;\quad T^*=T_w;\quad t^*=\frac{a}{v^*};\quad v^*=\sqrt{\frac{\varrho_0 gRT_w a}{\varrho_p L}},$$

where ρ_0 is the density of the gas in a state of equilibrium.

After transformations and a transition to dimensionless quantities, using the formulas

$$\bar{x}=\frac{x}{l^*};\quad \bar{T}_1=\frac{T_1}{T^*};\quad \bar{T}_2=\frac{T_2}{T^*};\quad \bar{t}=\frac{t}{t^*};\quad \bar{v}=\frac{v}{v^*}$$

we obtain a calculating system of differential equations in the form

$$\frac{d\bar{T}_1}{d\bar{t}}=\mathrm{St}\,\varkappa(1-\bar{T}_1)+\frac{\mathrm{St}\,\varkappa}{S}(\bar{T}-\bar{T}_1)(1-\bar{x})+\frac{(\varkappa-1)\bar{T}_1\bar{v}}{1-\bar{x}};\tag{9}$$

$$\frac{d\bar{T}_2}{d\bar{t}}=\mathrm{St}\,\varkappa(1-\bar{T}_2)+\frac{\mathrm{St}\,\varkappa(\bar{T}-\bar{T}_2)(1+\bar{x})}{S}-\frac{(\varkappa-1)\bar{T}_2\bar{v}}{1+\bar{x}};\tag{10}$$

$$\frac{d\bar{v}}{d\bar{t}}=\mathrm{N}\bar{v}+\frac{\bar{T}_2}{1+\bar{x}}-\frac{\bar{T}_1}{1-\bar{x}};\quad \frac{d\bar{x}}{d\bar{t}}=\bar{v}.\tag{11), (12}$$

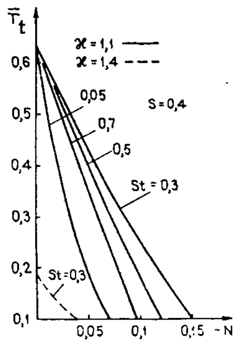


Fig. 2

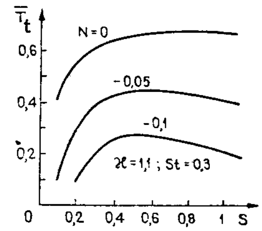


Fig. 3

Fig. 2. Dependence of the threshold ratio of the temperatures $\overline{T}_t$ on the parameter of MHD-interaction, N.

Fig. 3. Dependence of the threshold ratio of the temperatures $\overline{T}_t$ on the geometric parameter S.

The system (9)-(12) contains the dimensionless determining parameters: the Stanton number $St = \alpha/\rho_0 c_p v^*$; the index of isentropicity $\varkappa = c_p/c_v$; the geometric parameter $S = S_1/S_2 a$; the parameter of MHD-interaction $N = \sigma B^2 a(\eta_e - 1)\rho_p v^*$; the temperature factor $\overline{T} = T'_w/T_w$.

In a state of equilibrium

$$\overline{x}=0; \quad \overline{v}=0; \quad \overline{T}_{1,2}=(\overline{T}+S)(1+S)^{-1}=\overline{T}_0. \tag{12a}$$

The latter condition means the equality of the heat fluxes, arriving through the hot, and leaving through the cold, walls of the gas cavity. Let us investigate the stability of the equilibrium state of the system with respect to Lyapunov. To this end, we expand the right-hand parts of (9)-(12) in a Taylor series at the point of equilibrium, and discard terms of a higher order of smallness.

The linearized systems of equations, written for the deviations of the quantities from their values in a position of equilibrium, is written in the form

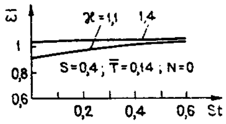
$$\begin{aligned}
&\frac{d(\Delta\overline{x})}{d\overline{t}}=\Delta\overline{v};\\
&\frac{d(\Delta\overline{T}_1)}{d\overline{t}}=\frac{\varkappa\,St}{S}(\overline{T}_0-\overline{T})\Delta\overline{x}-\frac{\varkappa\,St(S+1)}{S}\Delta\overline{T}_1+(\varkappa-1)\overline{T}_0\Delta\overline{v};\\
&\frac{d(\Delta\overline{T}_2)}{d\overline{t}}=\frac{\varkappa\,St}{S}(\overline{T}-\overline{T}_0)\Delta\overline{x}-\frac{\varkappa\,St(S+1)}{S}\Delta\overline{T}_2-(\varkappa-1)\overline{T}_0\Delta\overline{v};\\
&\frac{d(\Delta\overline{v})}{d\overline{t}}=-2\overline{T}_0\Delta\overline{x}-\Delta\overline{T}_1+\Delta\overline{T}_2+N\Delta\overline{v}.
\end{aligned} \tag{13}$$

Its solution is composed of combinations of functions of the form $e^{q\overline{t}}$, where q is the solution of the characteristic equation. In the case under consideration

$$q^4+a_1q^3+a_2q^2+a_3q+a_4=0, \tag{14}$$

where

$$\begin{aligned}
&a_1=2(S+1)\varkappa\,St\,S^{-1}-N;\\
&a_2=-2N(S+1)\varkappa\,St\,S^{-1}+2\varkappa(S+\overline{T})(1+S)^{-1}+[\varkappa(S+1)St\,S^{-1}]^2;\\
&a_3=4(S+\overline{T})\varkappa\,St\,S^{-1}+2\varkappa\,St(1-\overline{T})(S+1)^{-1}\\
&\quad -N[\varkappa(S+1)St\,S^{-1}]^2+2\varkappa\,St(\varkappa-1)(S+\overline{T})S^{-1};\\
&a_4=2(\varkappa\,St)^2(2S^{-1}+1+\overline{T}S^{-2}).
\end{aligned}$$



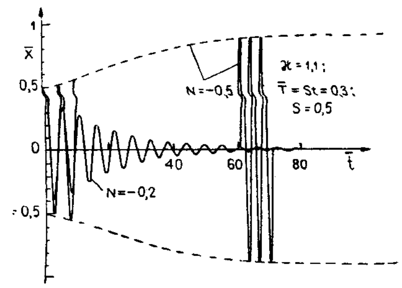


Fig. 4

Fig. 5

Fig. 4. Dependence of dimensionless frequency of vibrations $\bar{\omega}$ on Stanton number St.

Fig. 5. Curves of the establishment of auto-oscillations in the region of unstable equilibrium (N = −0.05) and the damping of the vibrations in the region of stable equilibrium (N = −0.2).

The quantity $q = r + i\bar{\omega}$ is a complex number. In the case of a rise in the vibrations (system unstable), $r > 0$; with their damping, $r < 0$. The value of $\bar{\omega}$ determines the frequency of the vibrations.

The sign of the real part of the roots of Eqs. (14) can be determined using the Liénard – Shipar conditions [2], in accordance with which, for all the roots of a polynomial to have a negative real part, it is necessary and sufficient that all its coefficients be positive and obey the inequalities

$$\Delta_{n-1}>0, \quad \Delta_{n-3}>0, \ldots$$

where

$$\Delta_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \ldots 0 \\ a_3 & a_2 & a_1 & a_0 & \ldots 0 \\ a_5 & a_4 & a_3 & a_2 & \ldots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots a_n \end{vmatrix},$$

n is the leading power of the polynomial.

In the case under consideration, the coefficients of the polynomial (14) are positive (with all values of the coefficients entering into them), $\Delta_1 = a_1 > 0$; therefore, the condition $\Delta_3 = 0$ (15) is the equation of the boundary of the region of stability. As an example, Fig. 2 shows the boundary of the region of stability, calculated using Eq. (14) in a Minsk-22 digital computer. The region below the reduced curve is the region of an unstable state of equilibrium.

As can be seen from Fig. 2, for the start of self-excitation, it is necessary that the ratio of the temperatures at the cold and hot walls $\bar{T}$ be less than some threshold value T_t.

This value decreases with a rise in the absolute value of the parameter of the MHD-interaction N, has a maximum with respect to the Stanton number St, and depends essentially on the adiabatic index $\varkappa$. This latter fact can be explained by the smaller increase in the temperature with a decrease in $\varkappa$ with compression of the gas by the piston in one of the cavities, so that the temperature of the hot source which is sufficient for there to be a supply of heat (and the start of auto-oscillation) may be lowered. The dependence of $\bar{T}_t$ on the geometric parameter S has a maximum with respect to S (Fig. 3). A direct numerical solution of Eq. (14) using a standard solution confirmed the boundaries found for the region of stability. The dimensionless frequency $\bar{\omega} = \omega a/v^*$ found with N = St = 0 is close to the adiabatic frequency $\bar{\omega}_{ad} = \bar{\omega} = \sqrt{2\varkappa T_0}$, and depends only weakly on the parameters N, St, $\varkappa$. Its dependence on the Stanton number is shown in Fig. 4. Further, the starting system of equations (9)-(12) was solved numerically by the Runge–Kutta method in a Minsk-22 digital computer. The initial conditions were the values of the variables in a position of equilibrium (12a), and the initial perturbation was given in the coordinate $\bar{x}$. The calculations confirmed the presence of auto-oscillations in the region of unstable equilibrium and their damping in the region of stable equilibrium (Fig. 5). The initial frequencies are close to those found using the equations of the first approximation; with the development of vibrations, the frequency rises somewhat.

The efficiency η is defined as the ratio of the energy given up in the external circuit in the period of vibrations T_V to the amount of heat supplied in this same period from the hot source, under conditions of fully established vibrations, i.e.,

$$\eta = \int_{t_0}^{t_0+T_k} \sigma B^2 v^2 \eta_e(\eta_e - 1) S_1 L \, dt \cdot \left[\int_{t_0}^{t_0+T_k} \alpha S_1 (T_w - T_1) \, dt \right]^{-1}. \tag{15}$$

For the example of Fig. 5, the efficiency $\simeq 3\%$.

Assuming (Fig. 4) $\bar{\omega} \simeq 1$, let us evaluate the frequency of the vibrations in terms of the scale velocity v^*:

$$\omega = \bar{\omega}\frac{v^*}{a} = \bar{\omega}\sqrt{\frac{\varrho_0 g R T_w}{\varrho_p L a}}. \tag{16}$$

Since gRT_W is close to the square of the speed of sound, and assuming $\alpha \approx L$, we obtain

$$\omega \approx \frac{c}{L}\sqrt{\frac{\varrho_0}{\varrho_p}}. \tag{17}$$

At the temperature of the hot wall, the speed of sound can be 500-600 m/sec, and the ratio of the densities of the metallic piston and the gas around 10^3. Summing up, for the frequency we obtain the evaluation $\omega \simeq 20/L$, i.e., with $L \approx 0.06$ m, $\omega \simeq 300$ rad/sec.

Thus, the simple device shown in the schematic diagram of Fig. 1 makes it possible to effect the conversion of heat into mechanical work or electrical energy as a result of the development of auto-oscillations. The examples calculated show that the efficiency of the device is not great. Further work is required in order to seek out conditions with an increased efficiency.

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