APPROXIMATE CALCULATION OF VELOCITY FIELDS IN CYLINDRICAL AND FLAT MHD CHANNELS

B. N. Siplivyi

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The calculation of velocity fields in cylindrical and flat MHD channels is solved. Approximate flow equations are used which were obtained by evaluation of terms in the Navier–Stokes equations. For a noninductive approximation, the calculations of velocity fields reduces to a solution of an integral Fredholm equation of the second kind. As an example, the velocity distribution in a flat MHD channel in a transverse nonuniform magnetic field is calculated.

1. We consider axisymmetric flow of a conducting fluid in a circular channel in an inhomogeneous magnetic field.

We assume the magnetic field is known and has only r and z components which are independent of α in a cylindrical coordinate system (r, α, z). Such a flow can be realized in the cylindrical channel shown in Fig. 1, for example.

The flow in a channel is described by the system of Navier–Stokes equations,

$$\rho\left(u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}\right)=-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}\right]+B_r\gamma(uB_r-vB_z);$$
$$\rho\left(u\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial r}\right)=-\frac{\partial p}{\partial r}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial^2 v}{\partial z^2}-\frac{v}{r^2}\right]+\gamma B_z(uB_r-vB_z); \quad (1.1)$$
$$\frac{1}{r}\frac{\partial}{\partial r}(rv)+\frac{\partial u}{\partial z}=0.$$

Here, u is the z component of the velocity U, v is the r component, the last terms in the first two equations are the projections of the volume electromagnetic force [jB] on the coordinate axes, γ is the conductivity of the fluid, and j is the density of the induced currents.

Following [1], we simplify the system of equations (1.1). We introduce the dimensionless variables

$$B_r=B^*_rB^0;\quad B_z=B^*_zB^0;\ u=u^*\tilde{U};\quad v=v^*\tilde{V};\quad z=Lz^*;\quad r=ar^*; \quad (1.2)$$
$$p=p^*\frac{\mu\tilde{U}L}{a^2};\quad \mathrm{Re}=\frac{\tilde{U}a}{\nu};$$

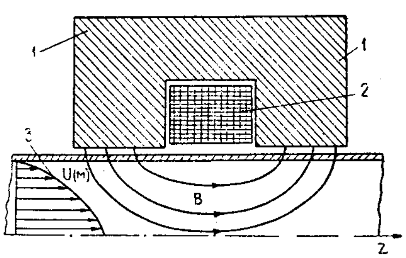

Fig. 1. Channel section. 1) Magnetic circuit; 2) magnetic coil; 3) velocity distribution.

$\tilde{U}$ and $\tilde{V}$ are typical longitudinal and transverse velocities, L is a typical longitudinal dimension which is equal to the longitudinal dimension of the magnetic system, a is a typical transverse dimension (a = R), B^0 is the induction of a typical magnetic field, and $\nu = \mu/\rho$. We set $\tilde{V}/\tilde{U} = \varepsilon_1$, and $a/L = \varepsilon_2$. From the equation of continuity we find that ε_1 and ε_2 are of the same order of magnitude: $\tilde{V}/\tilde{U} \sim a/L = \varepsilon$. After substitution of Eqs. (1.2) in (1.1) We obtain

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$$\mathrm{Re}\,\varepsilon\left(u^*\frac{\partial u^*}{\partial z^*}+v^*\frac{\partial u^*}{\partial r^*}\right)=-\frac{\partial p^*}{\partial z^*}+\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u^*}{\partial r^*}\right)+\varepsilon^2\frac{\partial^2 u^*}{\partial z^{*2}}+\mathrm{Ha}^2u^*B^*{}_r-\mathrm{Ha}^2\varepsilon v^*B^*{}_rB^*{}_z;$$

$$\mathrm{Re}\,\varepsilon^3\left(u^*\frac{\partial v^*}{\partial z^*}+v^*\frac{\partial v^*}{\partial r^*}\right)=-\frac{\partial p^*}{\partial r^*}+\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial v^*}{\partial r^*}\right)\varepsilon^2+\varepsilon^4\frac{\partial^2 v^*}{\partial z^{*2}}+\mathrm{Ha}^2\varepsilon u^*B^*{}_rB^*{}_z-\mathrm{Ha}^2\varepsilon^2B^*{}_z{}^2v^*; \tag{1.3}$$

$$\frac{\partial u^*}{\partial z^*}+\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^*v^*)=0;$$

$\mathrm{Ha}^2 = B^\circ\alpha\gamma/\mu$ is the Hartmann number.

We limit ourselves to the class of flows for which the condition $\varepsilon \ll 1$ is satisfied. In this case, one can neglect in Eqs. (1.3) terms of smallness ε^3 and ε^4 in comparison with the other terms. Converting to the dimensional form for the equations and linearizing them by the method of Oseen, we obtain the equations

$$\frac{\partial u}{\partial z}=-\frac{1}{\varrho}\frac{\partial p}{\partial z}+\nu\Delta u+\frac{1}{\varrho}jB_r;$$

$$\frac{1}{\varrho}\frac{\partial p}{\partial r}=\nu\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rv)+\frac{1}{\varrho}jB_z;\quad \frac{1}{r}\frac{\partial}{\partial r}(rv)+\frac{\partial u}{\partial z}=0, \tag{1.4}$$

where $\Delta \equiv (1/r)(\partial/\partial r)r[(\partial/\partial r) + (\partial^2/\partial z^2)]$.

Introducing the velocity and pressure perturbations $u = U_0 + u_1$, $p = p_0 \pm p_1$, and the new dimensionless variables $u_1 = \tilde{U}u_1^*$, $v_1 = \tilde{V}v^*{}_1/\sqrt{2}$; $r = \alpha r^*$, $z = \alpha z^*/\sqrt{2}$, $p_1 = p_1^*\rho\tilde{U}^2/\mathrm{Re}$, $\mathrm{Re} = \tilde{U}\alpha/\nu\sqrt{2}$, $B_r = B^\circ B_r^*$, $B_z = B^\circ B_z^*$, and $\mathrm{Ha}^2 = B^\circ\alpha\gamma/\mu$, we obtain a system of equations for the dimensionless velocity perturbations (we keep the notation for the dimensionless quantities the same as for the dimensional quantities):

$$\mathrm{Re}\frac{\partial u_1}{\partial z}=-\frac{\partial p_1}{\partial z}+\frac{1}{2}\frac{\partial^2 u_1}{\partial z^2}+\frac{\partial^2 u_1}{\partial r^2}+\frac{1}{r}\frac{\partial u_1}{\partial r}-\mathrm{Ha}^2jB_r;$$

$$\frac{\partial p_1}{\partial r}=\frac{1}{2}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rv_1)+\mathrm{Ha}^2jB_z; \tag{1.5}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_1)+\frac{\partial u_1}{\partial z}=0;\quad \int_0^1 ru_1\,dr=0.$$

Such flows can be realized in practice in MHD channels with a weak magnetohydrodynamic interaction.

The last relation in (1.5) follows from the constancy of the flow rate through any cross section of the pipe.

We eliminate the pressure p_1 from the system (1.5). To do this, we integrate the second equation in (1.5),

$$p_1=-\frac{1}{2}\frac{\partial u_1}{\partial z}+F(z)+\mathrm{Ha}^2\int_0^r jB_z\,dr. \tag{1.6}$$

After substitution in the first equation in (1.5), we obtain

$$\mathrm{Re}\frac{\partial u_1}{\partial z}=-\frac{dF}{dz}+\Delta u_1-\mathrm{Ha}^2\int_0^r\frac{\partial}{\partial z}(jB_z)\,dr-\mathrm{Ha}^2jB_r. \tag{1.7}$$

Multiplying Eq. (1.7) by r, integrating over the cross section, and taking the constant flow rate into account, we find

$$\frac{dF}{dz}=2\left.\frac{\partial u_1}{\partial r}\right|_1-\Psi(z)\mathrm{Ha}^2, \tag{1.8}$$

where

$$\Psi(z)=2\left[\int_0^1 r\int_0^r\frac{\partial}{\partial z}(jB_z)\,dr\,dr+\int_0^1 rjB_r\,dr\right].$$

Substituting Eq. (1.8) into Eq. (1.7), we obtain

$$\Delta u_1-\mathrm{Re}\frac{\partial u_1}{\partial z}-2\left.\frac{\partial u_1}{\partial r}\right|_1=-j, \tag{1.9}$$

where

$$f=-\mathrm{Ha}^2\left[\int_0^r \frac{\partial}{\partial z}(jB_z)\,\mathrm{d}r+jB_r-\Psi(z)\right],\quad \Delta\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

The boundary conditions for Eq. (1.9) have the form

$$u_1(1,z)=0;\quad u_1(r,z)\to 0 \ \text{ when } \ z\to\pm\infty;\quad u_1(0,z)<\infty. \tag{1.10}$$

We seek a solution of Eq. (1.9) in the form of a series,

$$u_1(r,z)=\sum_{n=1}^{\infty}u_n(z)\cdot R_n(r), \tag{1.11}$$

where the $R_n(r)$ are eigenfunctions of the problem

$$R_n''+\frac{1}{r}R_n'+\lambda_n R_n=2R_n'(1);\quad R_n(1)=0;\quad R_n(0)<\infty. \tag{1.12}$$

We determine these eigenfunctions. When $\lambda = 0$, we find the zeroth eigenfunction by simple integration:

$$R_0(r)=A(1-r^2).$$

We determine the constant A from the last condition in (1.5): A = 0. The remaining eigenfunctions are of the form

$$R_n(r)=J_0(\beta_n r)+c;\quad \lambda_n=\beta_n^2.$$

We determine c by satisfying the boundary conditions in (1.12):

$$R_n(r)=J_0(\beta_n r)-J_0(\beta_n). \tag{1.13}$$

Substituting Eq. (1.13) into the equation in (1.12), we find that β_n is a root of the transcendental equation $J_2(\beta_n) = 0$. We normalize the eigenfunctions:

$$R_n(r)=\frac{\sqrt{2}}{J_1(\beta_n)}[J_0(\beta_n r)-J_0(\beta_n)]. \tag{1.14}$$

The eigenfunctions (1.14) possess the following properties:

1) $\int_0^1 rR_n(r)\,\mathrm{d}r=0,\quad n\neq 0;$

2) $\int_0^1 rR_n(r)\cdot R_k(r)\,\mathrm{d}r=\begin{cases}0, & n\neq k;\\ 1, & n=k;\end{cases}\quad n\neq 0,\ k\neq 0;$

3) the system of eigenfunctions (1.14) is complete in the class of differentiable functions satisfying the condition

$$\int_0^1 rf(r)\,\mathrm{d}r=0. \tag{1.15}$$

The first and second properties can be checked by direct substitution. The third property of the eigenfunctions is demonstrated by reduction of the boundary-value problem (1.12) to an integral equation with a symmetric kernel (for brevity, proof is omitted).

We return to the solution of Eq. (1.9). It is obvious that the function f satisfies the condition (1.15) and can be expanded in a series in terms of the eigenfunctions (1.14),

$$f(r,z)=\sum_{n=1}^{\infty}f_n(z)\cdot R_n(r), \tag{1.16}$$

where

$$f_n(z)=\frac{\sqrt{2}}{J_1(\beta_n)}\int_0^1 rf(r,z)\,[J_0(\beta_n r)-J_0(\beta_n)]\,\mathrm{d}r. \tag{1.17}$$

The expression for f_n can be simplified by taking Eq. (1.15) into account:

$$f_n(z)=\frac{\sqrt{2}}{J_1(\beta_n)}\int_0^1 rf(r,z)\cdot J_0(\beta_n r)\,\mathrm{d}r. \tag{1.18}$$

Substituting Eqs. (1.16) and (1.11) into Eq. (1.9), we obtain an equation for the determination of $u_n(z)$:

$$u_n'' - \mathrm{Re}\, u_n' - \beta_n^2 u_n = -f_n. \tag{1.19}$$

We define the Green's function $\omega_n(z, \zeta)$ of the operator $L(u_n) = u'' - \mathrm{Re}\, u_n' - \beta_n^2 u_n$ in the following manner [2]:

$$\begin{gathered}\frac{d^2}{dz^2}\omega_n(z,\zeta) - \mathrm{Re}\frac{d}{dz}\omega_n(z,\zeta) - \beta_n^2\omega_n(z,\zeta) = 0;\\ \omega_n(-\infty,\zeta) = \omega_n(\infty,\zeta) = 0;\\ \frac{d}{dz}\omega_n(z,\zeta)\Big|_{z=\zeta-0}^{z=\zeta+0} = -1.\end{gathered} \tag{1.20}$$

Functions satisfying the conditions (1.20) have the form

$$\omega_n(z,\zeta) = \begin{cases}\dfrac{\exp[-k_{1n}(z-\zeta)]}{2k_n}, & z \geqslant \zeta;\\ \dfrac{\exp[-k_{2n}(\zeta-z)]}{2k_n}, & z \leqslant \zeta.\end{cases} \tag{1.21}$$

where $k_n = \sqrt{\mathrm{Re}^2/4 + \beta_n^2}$, $k_{1n} = k_n + \mathrm{Re}/2$, $k_{2n} = k_n - \mathrm{Re}/2$. By means of the Green's function, the solution of Eq. (1.19) can be represented in the form

$$u_n(z) = \int_{-\infty}^{\infty} f_n(\zeta)\cdot\omega_n(z,\zeta)\,d\zeta. \tag{1.22}$$

Substituting Eqs. (1.14) and (1.22) in Eq. (1.11), we obtain

$$u_1(r,z) = \sum_{n=1}^{\infty}\left\{\int_{-\infty}^{\infty} f_n(\zeta)\cdot\omega_n(z,\zeta)\,d\zeta\cdot\frac{\sqrt{2}}{J_1(\beta_n)}[J_0(\beta_n r) - J_0(\beta_n)]\right\}. \tag{1.23}$$

Interchanging the order of summation and integration in Eq. (1.23) and taking Eq. (1.18) into consideration, we obtain a final expression for the velocity perturbation

$$u_1(M) = \int_S f(N)\cdot K(M,N)\,dS_N, \tag{1.24}$$

where

$$K(M,N) = 2r_N\sum_{n=1}^{\infty}\frac{J_0(\beta_n r_N)}{J_1^2(\beta_n)}[J_0(\beta_n r_M) - J_0(\beta_n)]\,\omega_n(z_M, z_N); \tag{1.25}$$

and S is the area of a meridional section of the channel.

Adding a Poiseuille distribution to the right side, we obtain an expression for the longitudinal velocity,

$$u(M) = \int_S f(N)\cdot K(M,N)\,dS_N + 2(1 - r_M^2). \tag{1.26}$$

Knowing u(M), one can determine $v_1(M)$ from the continuity equation:

$$v_1(M) = \frac{1}{r_M}\int_0^r r\frac{\partial u_1}{\partial z}\,dr. \tag{1.27}$$

The pressure distribution can be determined from Eqs. (1.6) and (1.8):

$$p(M) = p_0 + 2\int\left(\frac{\partial u}{\partial r}\right)_1 dz - \mathrm{Ha}^2\int\Psi(z)\,dz - \frac{1}{2}\frac{\partial u}{\partial z} + \mathrm{Ha}^2\int_0^r jB_z\,dr. \tag{1.28}$$

Note that it is necessary to know the distribution of the magnetic field and of the induced current along a meridional section of the channel in order to use Eqs. (1.26), (1.27), and (1.28), for arbitrary Re_m. When $\mathrm{Re}_m \ll 1$ (noninductive approximation), the calculation reduces to the solution of the system of equations

$$\delta(M) = u(M)\cdot B_r(M);\quad u(M) = \int_S f(N)\cdot K(M,N)\,dS_N. \tag{1.29}$$

2. We consider flow in an annular channel formed by two coaxial cylinders with radii r_0 and r_1 in an inhomogeneous magnetic field $\mathbf{B}(M) = \mathbf{e}_r B_r(r, z) + \mathbf{e}_z B_z(r, z)$. In this case the approximate system of Navier–Stokes equations (1.1) reduces to

$$\Delta u_1 - \mathrm{Re}\frac{\partial u_1}{\partial z} - \frac{2}{r_1^2 - r_0^2}\left(r\frac{\partial u_1}{\partial r}\Big|_{r_0}^{r_1}\right) = -f, \tag{2.1}$$

where u_1 is the perturbation of the longitudinal velocity:

$$f = -\mathrm{Ha}^2 \left\{ \int_{r_0}^{r} \frac{\partial}{\partial z}(jB_z)\,\mathrm{d}r + jB_r - \int_{r_0}^{r_1} r \int_{r_0}^{r} \frac{\partial}{\partial z}(jB_z)\,\mathrm{d}r\mathrm{d}r - \int_{r_0}^{r_1} rjB_r\mathrm{d}r \right\}. \quad (2.2)$$

The boundary conditions for u_1 are the following:

$$u_1|_{r_0} = u_1|_{r_1} = 0, \quad u_1(+\infty) = u_1(-\infty) = 0. \quad (2.3)$$

We write the solution of Eq. (2.1) in the form

$$u_1(r, z) = \sum_{n=1}^{\infty} u_n(z) \cdot R_n(r), \quad (2.4)$$

where the $R_n(r)$ are the eigenfunctions of the problem

$$R_n'' + \frac{1}{r} R_n' + \beta_n^2 R_n = \frac{2}{r_1^2 - r_0^2} (rR_n') \Big|_{r_0}^{r_1}; \quad R_n(r_0) = R_n(r_1) = 0. \quad (2.5)$$

The eigenfunctions are of the form

$$R_n(r) = c_1 J_0(\beta_n r) + c_2 N_0(\beta_n r) - A. \quad (2.6)$$

We determine c_1 and c_2 by satisfying the boundary conditions:

$$c_1 = \frac{A[N_0(\beta_n r_1) - N_0(\beta_n r_0)]}{D}, \quad c_2 = \frac{A[J_0(\beta_n r_0) - J_0(\beta_n r_1)]}{D}, \quad (2.7)$$

where

$$D = J_0(\beta_n r_0) \cdot N_0(\beta_n r_1) - N_0(\beta_n r_0) \cdot J_0(\beta_n r_1).$$

Substituting Eq. (2.6) into Eq. (2.5), we find that β_n is a root of the transcendental equation

$$\beta_n D = \frac{2}{r_1^2 - r_0^2} \left\{ \frac{4}{\pi \beta_n} + r_1 [J_0(\beta_n r_0) \cdot N_1(\beta_n r_1) - J_1(\beta_n r_1) \cdot N_0(\beta_n r_0)] + \right.$$
$$\left. + r_0 [J_0(\beta_n r_1) \cdot N_1(\beta_n r_0) - J_1(\beta_n r_0) \cdot N_0(\beta_n r_1)] \right\}. \quad (2.8)$$

Substituting c_1 and c_2 into Eq. (2.6), we obtain

$$R_n(r) = \frac{A}{D} \{ [N_0(\beta_n r_1) - N_0(\beta_n r_0)] J_0(\beta_n r) [J_0(\beta_n r_0) - J_0(\beta_n r_1)] N_0(\beta_n r) - D \}. \quad (2.9)$$

We determine the constant A/D from the normalization condition $\int_{r_0}^{r_1} r R_n^2(r)\,\mathrm{d}r = 1$:

$$\frac{A}{D} = \frac{2}{\begin{array}{c} r_1^2 \{ [J_0(\beta_n r_1) - J_0(\beta_n r_0)] N_0(\beta_n r_1) + [N_0(\beta_n r_0) - N_0(\beta_n r_1)] J_0(\beta_n r_0) \}^2 - \\ - r_0^2 \{ [J_0(\beta_n r_1) - J_0(\beta_n r_0)] N_0(\beta_n r_0) + [N_0(\beta_n r_0) - N_0(\beta_n r_1)] J_0(\beta_n r_0) \}^2 \end{array}}.$$

The eigenfunctions (2.9) have the following properties:

1) $\int_{r_0}^{r_1} r R_n(r)\,\mathrm{d}r = 0, \quad n \neq 0;$

2) $\int_{r_0}^{r_1} r R_n(r) \cdot R_k(r)\,\mathrm{d}r = \begin{cases} 0, & n \neq k; \\ 1, & n = k; \end{cases} \quad n \neq 0, \; k \neq 0;$

3) the system of eigenfunctions is complete in the class of functions for which $\int_{r_0}^{r_1} r f(r)\,\mathrm{d}r = 0$.

Expanding $f(r, z)$ in a series in terms of the eigenfunctions (2.9) and then substituting in Eq. (2.1), we obtain an equation for u_n:

$$u_n'' - \mathrm{Re}\, u_n' - \beta_n^2 u_n = -f_n(z) = -\int_{r_0}^{r_1} r f(r, z) \cdot R_n(r)\,\mathrm{d}r. \quad (2.10)$$

The solution of Eq. (2.10) was found in Sec. 1. Substituting it in Eq. (2.4), we obtain an expression for the perturbation of the longitudinal velocity:

$$u_1(r, z) = \sum_{n=1}^{\infty} R_n(r) \int_{-\infty}^{\infty} f_n(\zeta) \cdot \omega_n(z, \zeta)\, d\zeta. \quad (2.11)$$

Interchanging the order of summation and integration and adding the velocity distribution when B = 0 to the right side, we obtain a final expression for the longitudinal velocity:

$$u(M) = \int_S f(N) \cdot K(M, N)\, dS_N + \frac{2}{r_1^2 + r_0^2 - (r_1^2 - r_0^2)/\ln(r_1/r_0)} \left[r_0^2 - r_M^2 + \frac{r_1^2 - r_0^2}{\ln(r_1/r_0)} \ln \frac{r_M}{r_0} \right], \quad (2.12)$$

where

$$K(M, N) = r_N \sum_{n=1}^{\infty} R_n(r_M) \cdot R_n(r_N) \cdot \omega_n(z_M, z_N). \quad (2.13)$$

In those cases where the conditions $B_z \ll B_r$, $B_r = B_r(z)/r$ are satisfied, the expression for f simplifies to

$$f(r, z) = -\mathrm{Ha}^2 \left[u \frac{B_r^2(z)}{r^2} + U \frac{B_r^2(z)}{r^2} - \int_{r_0}^{r_1} (u + U) \frac{B_r^2(z)}{r^2}\, dr \right], \quad (2.14)$$

where U is the velocity distribution in the channel when B = 0.

Substituting Eq. (2.14) into Eq. (2.11) and taking the properties of the eigenfunctions $R_n(r)$ into consideration, we obtain after several simple transformations an integral Fredholm equation of the second kind with respect to the perturbation of longitudinal velocity:

$$u_1(M) = -\mathrm{Ha}^2 \int_S u_1(N) \cdot K_2(M, N)\, dS_N + f_1(M), \quad (2.15)$$

where

$$K_2(M, N) = \frac{B_r^2(z)}{r_N^2} \sum_{n=1}^{\infty} R_n(r_M) \cdot R_n(r_N) \cdot \omega_n(z_M, z_N);$$

$$f_1(M) = -\mathrm{Ha}^2 \int_S U(r_N) \cdot K_2(M, N)\, dS_N.$$

The integral equation (2.15) can be used to determine velocity distribution in an annular channel in a noninductive approximation ($\mathrm{Re_m} \ll 1$).

3. We now consider plane flow between two parallel infinite walls in a transverse magnetic field $\mathbf{B}(M) = \mathbf{e}_y B_y(x, y) + \mathbf{e}_x B_x(x, y)$ ($\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors in a Cartesian coordinate system x, y, z).

We assume the field B and $\mathbf{j} = \mathbf{e}_z j_z(x, y)$ are known, $\mathbf{U}(M) = \mathbf{e}_x u(x, y) + \mathbf{e}_y v(x, y)$, and flow is symmetric with respect to the plane y = 0.

We consider the class of flows for which the condition $(v/u) \sim (h/L) \ll 1$ is satisfied, where h is the half-height of the channel and L is a typical longitudinal dimension. In this case, the system of Navier–Stokes equations describing plane flow reduces to

$$\Delta u_1 - \mathrm{Re} \frac{\partial u_1}{\partial x} - \left.\frac{\partial u_1}{\partial y}\right|_1 = -f, \quad (3.1)$$

where

$$f = -\mathrm{Ha}^2 \left[j_z B_y + \int_0^y \frac{\partial}{\partial x}(j_z B_x)\, dy - \int_0^1 \int_0^y \frac{\partial}{\partial x}(j_z B_x)\, dy\, dy - \int_0^1 j_z B_y\, dy \right]; \quad (3.2)$$

u_1 is the perturbation of the longitudinal velocity.

The boundary conditions have the form $u_1(0) = 0$, $u_1(+\infty) = u_1(-\infty) = 0$. The solution is obtained in a manner similar to that for the solution for a circular pipe and is

$$u_1(M) = \int_S f(N) \cdot K_3(M, N)\, dS_N, \quad (3.3)$$

where

$$K_3(M, N) = \sum_{n=1}^{\infty} \frac{2 \cos(\alpha_n y_N)}{\sin^2 \alpha_n} [\cos(\alpha_n y_M) - \cos \alpha_n] \omega_n(z_M, z_N) \quad (3.4)$$

is a root of the equation $\tan \mu = \mu$, and

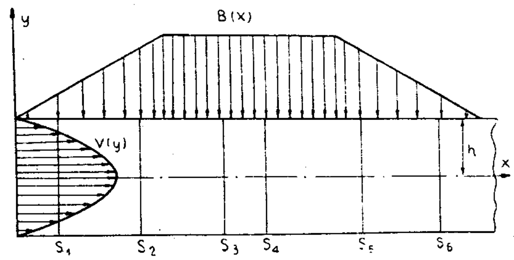

Fig. 2

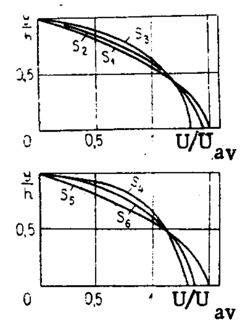

Fig. 3

Fig. 2. Section of a flat MHD channel by the plane z = const. Velocity distribution in the initial section and magnetic field along the channel; S_1, S_2, S_3, S_4, S_5, and S_6 are sections of the channel by planes for which the velocity profiles are shown.

Fig. 3. Velocity profiles at the sections S_1–S_6 in a flat MHD channel for Re = 10 and Ha = 4.

$$\omega_n(z_M, z_N) = \begin{cases} \dfrac{\exp k_{1n}(z_M - z_N)}{2k_n}, & z_M \leqslant z_N; \\ \dfrac{\exp k_{2n}(z_N - z_M)}{2k_n}, & z_M \geqslant z_N. \end{cases}$$

If we consider flows for which the condition $B_x \ll B_y$ is satisfied, one can replace $B_y(x, y)$ in Eq. (3.2) by its average value over a cross section $B_y(x)$. Taking Ohm's law and the conservation of flow rate into consideration, the expression for f can be rewritten as

$$f = -\mathrm{Ha}^2[{}^3/_2(1-y^2) + u_1]B_y{}^2 + \mathrm{Ha}^2 B_y{}^2. \tag{3.5}$$

Substituting Eq. (3.5) into Eq. (3.3), we obtain an integral equation of the second kind with respect to the perturbation u_1 of the longitudinal velocity:

$$u_1(M) + \mathrm{Ha}^2 \int_S u_1(N) \cdot B_y(x_N) \cdot K_3(M, N)\, dS_N = F(M), \tag{3.6}$$

where

$$F(M) = \mathrm{Ha}^2 \int_S B_y{}^2(x_N) \cdot K_3(M, N)\, dS_N. \tag{3.7}$$

It is sufficient to know the field distribution $B_y(x)$ in order to solve Eq. (3.6).

In conclusion, we make some remarks about the series appearing in Eqs. (1.25), (2.13), and (3.4). One can show that these series converge uniformly when $M \neq N$, and when $M = N$ the functions defined by Eqs. (1.25), (2.13), and (3.4) have a logarithmic singularity (for brevity, proof is omitted).

As an illustration, the flow in a plane channel in a nonuniform transverse magnetic field was calculated in the noninductive approximation (Fig. 2). The integral equation (3.6) was solved by the moments method [3] on a computer. The calculated velocity distributions u are shown in Fig. 3.

LITERATURE CITED

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