DEVELOPMENT OF INITIAL PERTURBATIONS IN AN ELECTRICALLY CONDUCTING LIQUID INITIALLY AT REST IN A TRAVELING MAGNETIC FIELD

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1. There exists a sequence of critical numbers (dimensionless flow rate [1] or magnetic Reynolds number [2]), in transition through which the homogeneous flow of an electrically conducting liquid in a traveling magnetic field becomes unstable to perturbations of a certain scale. The corresponding supercritical motions can be obtained from the equations of steady-state approximations [1-3].

However, the problem of realizability of any supercritical motion cannot be solved in the steady-state formulation. One of the possible methods of solving this problem is to investigate the development of the corresponding initial perturbations, and this requires the solution of nonlinear nonstationary equations with initial conditions.

2. As the initial system of equations we shall make use of the system presented in [2] describing nonstationary plane-parallel motion in a traveling field.

For simplicity we shall restrict ourselves to the case of "closed" channels (the flow rate is fixed and equal to zero) and constancy of the phase current (no perturbations of the external field).

Assuming the flow to be turbulent and introducing the dimensionless quantities, we obtain

$$\frac{\partial \bar{v}}{\partial \bar{t}} = -\bar{c} - \frac{\lambda \bar{v}^2}{\bar{\delta}^2} \operatorname{sign} \bar{v} + \mathrm{St}\, \bar{h}_a; \tag{1}$$

$$\frac{\partial^2 \bar{h}_r}{\partial \bar{y}^2} - \mathrm{Rm}_s \frac{\partial \bar{h}_r}{\partial \bar{t}} + \bar{h}_a - \bar{v}(1 + \bar{h}_a) - \bar{h}_r = 0; \tag{2}$$

$$\frac{\partial^2 \bar{h}_a}{\partial \bar{y}^2} - \mathrm{Rm}_s \frac{\partial \bar{h}_a}{\partial \bar{t}} + \mathrm{Rm}_s^2 \bar{h}_r (\bar{v} - 1) - \bar{v} - \bar{h}_a = 0. \tag{3}$$

Equation (1) is the equation of motion, and (2) and (3) are the real and imaginary parts of the induction equation written in complex form; $\bar{v}$, $\bar{h}_a$, $\bar{h}_r$ are the dimensionless perturbations of the velocity and the components of the complex amplitude of the intrinsic field intensity. The dimensionless quantities are introduced according to the formulas

$$\bar{t} = t\alpha v_s; \quad \bar{v} = v'/v_s; \quad \bar{y} = y\alpha; \quad \bar{h}_a = h'_a/h_{a0};$$

$$\bar{h}_r = h'_r/h_{r0}; \quad \bar{\delta} = 4\delta\alpha; \quad \alpha = \pi/\tau_0;$$

$$\mathrm{St} = \frac{\sigma B_1^2}{2\rho\alpha v_s}; \quad B_1^2 = \frac{\mu_0 H_{m0}^2}{1 + \mathrm{Rm}_s^2}; \quad \mathrm{Rm}_s = \frac{\sigma\mu v_s}{\alpha}; \quad \bar{c} = \frac{\partial p'/\partial x}{\rho\alpha v_s^2}.$$

Here the "prime" denotes perturbations, the index "0" refers to the unperturbed flow, St is the Stewart number constructed from the induction of the total magnetic field, Rm_s is the magnetic Reynolds number constructed from the velocity of the traveling field v_s, and τ_0 is the pole division. Restricting ourselves to the case of a coaxial channel we introduce the dimensionless width of the flow; $\bar{T} = 2\pi/\bar{\lambda}_1$, where $\bar{\lambda}_1 = \tau/\pi R$ is a geometrical parameter.

The system of equations (1)-(3) satisfies the periodic boundary conditions

$$\bar{h}_a\big|_{\pi/\bar{\lambda}_1} = \bar{h}_a\big|_{-\pi/\bar{\lambda}_1}; \quad \bar{h}_r\big|_{\pi/\bar{\lambda}_1} = \bar{h}_r\big|_{-\pi/\bar{\lambda}_1}$$
$$\frac{\partial \bar{h}_a}{\partial \bar{y}}\bigg|_{\pi/\bar{\lambda}_1} = \frac{\partial \bar{h}_a}{\partial \bar{y}}\bigg|_{-\pi/\bar{\lambda}_1}; \quad \frac{\partial \bar{h}_r}{\partial \bar{y}}\bigg|_{\pi/\bar{\lambda}_1} = \frac{\partial \bar{h}_r}{\partial \bar{y}}\bigg|_{-\pi/\bar{\lambda}_1} \tag{4}$$

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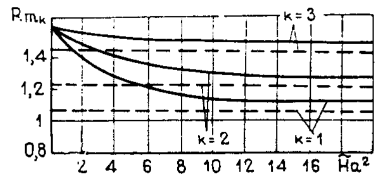


Fig. 1. Dependence of the critical magnetic Reynolds number Rm_k on the Hartmann number Ha^2. Continuous lines: friction taken into consideration; dashed lines: friction disregarded.

If we restrict ourselves to the case of initial perturbations which are even in $\bar{y}$ (initial conditions), then from the invariance of system (1)-(3) to the substitution of $\bar{y}$ by $-\bar{y}$, it follows that its solutions are even functions. Since the derivatives of even functions are odd functions, we have

$$\frac{\partial \bar{h}_a}{\partial \bar{y}}\Big|_{\pi/\bar{\lambda}_1} = -\frac{\partial \bar{h}_a}{\partial \bar{y}}\Big|_{-\pi/\bar{\lambda}_1}; \qquad \frac{\partial \bar{h}_r}{\partial \bar{y}}\Big|_{\pi/\bar{\lambda}_1} = -\frac{\partial \bar{h}_r}{\partial \bar{y}}\Big|_{-\pi/\bar{\lambda}_1}. \tag{5}$$

Comparing (4) and (5) we get

$$\frac{\partial \bar{h}_a}{\partial \bar{y}}\Big|_{\pi/\bar{\lambda}_1} = \frac{\partial \bar{h}_a}{\partial \bar{y}}\Big|_{-\pi/\bar{\lambda}_1} = 0; \qquad \frac{\partial \bar{h}_r}{\partial \bar{y}}\Big|_{\pi/\bar{\lambda}_1} = \frac{\partial \bar{h}_r}{\partial \bar{y}}\Big|_{-\pi/\bar{\lambda}_1} = 0. \tag{6}$$

From the even character of the solutions of the system of equations (1)-(3) it follows that

$$\frac{\partial \bar{h}_a}{\partial \bar{y}}\Big|_0 = \frac{\partial \bar{h}_r}{\partial \bar{y}}\Big|_0 = 0. \tag{7}$$

Now, making use of (6) and (7), we obtain the boundary conditions for the half-width of the channel:

$$\frac{\partial \bar{h}_a}{\partial \bar{y}}\Big|_0 = \frac{\partial \bar{h}_a}{\partial \bar{y}}\Big|_{\pi/\bar{\lambda}_1} = \frac{\partial \bar{h}_r}{\partial \bar{y}}\Big|_0 = \frac{\partial \bar{h}_r}{\partial \bar{y}}\Big|_{\pi/\bar{\lambda}_1} = 0. \tag{8}$$

We shall specify the initial conditions in the form

$$\bar{v} = \bar{v}_0 \cos k\bar{\lambda}_1\bar{y}; \qquad \bar{h}_a = \bar{h}_r = 0, \tag{9}$$

where k is an integer.

Since in a "closed" channel for even $\bar{v}$ we must have

$$\int_0^{\pi/\bar{\lambda}_1} \bar{v}\, d\bar{y} = \frac{1}{2}\int_0^{2\pi/\bar{\lambda}_1} \bar{v}\, d\bar{y} = 0,$$

integrating (1) over the half-width $\pi/\bar{\lambda}_1$, we obtain the dimensionless pressure gradient $\bar{c}$ in the form

$$\bar{c} = \int_0^{\pi/\bar{\lambda}_1} \frac{\bar{\lambda}_1}{\pi}\left(\mathrm{St}\,\bar{h}_a - \frac{\lambda\bar{v}^2}{\bar{\delta}}\,\mathrm{sign}\,\bar{v}\right) d\bar{y}. \tag{10}$$

The system of equations (1)-(3), (10) with boundary conditions (8) and initial condition (9) enables us to compute the development of supercritical motions.

3. Below we shall need the instability conditions for the investigated flow. These conditions are obtained in [1], where dimensionless flow rates are taken as the critical numbers; this is convenient for constructing the instability limits in the flow-thrust plane. As in [2], here we start from considerations of physical clarity and take the magnetic Reynolds number as the critical number

$$\mathrm{Rm} = \mu\sigma(v_s - v_0)\,\tau_0/\pi.$$

In the notations of this work the instability condition [1] is written in the form

$$2\,|\,\mathrm{Rm}_s - \mathrm{Rm}_k|(\mathrm{Rm}_k^2 + \mathrm{Rm}_{k0}^2)(1 + \mathrm{Rm}_k^2) + \widetilde{\mathrm{Ha}}^2 \cdot \mathrm{Rm}_s^2(\mathrm{Rm}_{k0}^2 - \mathrm{Rm}_k^2) = 0, \tag{11}$$

where $\mathrm{Rm}_{k0} = 1 + k^2\lambda_1^2$ is the critical Reynolds number in the absence of friction; $\widetilde{\mathrm{Ha}}^2 = 2\mu_0\alpha H_{m0}^2\delta/\rho\lambda v_s^2$ is the Hartmann number. An example of dependence (11), illustrating the effect of friction, is shown in Fig. 1. In certain cases it is more convenient to use the condition

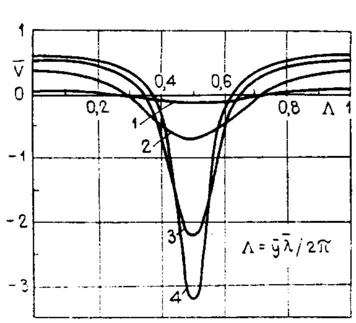


Fig. 2

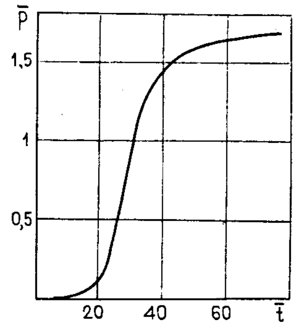


Fig. 3

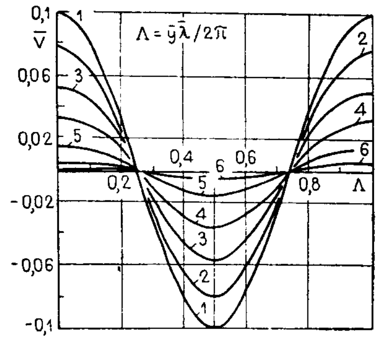


Fig. 4

Fig. 2. Development of velocity perturbations in the absence of friction: 1) $\bar{t} = 0$; 2) $\bar{t} = 26$; 3) $\bar{t} = 42$; 4) $\bar{t} = 77$; $\bar{v}_0 = 0.05 \cos \bar{\lambda}\bar{y}$; $\mathrm{Rm_s} = 2$; $\mathrm{St} = 15$; $\bar{\lambda}/\bar{\delta} = 0$; $\bar{\lambda} = \pi/2$.

Fig. 3. Development of pressure perturbations in the absence of friction.

Fig. 4. Damping of velocity perturbations for $\mathrm{Rm} < \mathrm{Rm}_1$: 1) $\bar{t} = 0$; 2) $\bar{t} = 61$; 3) $\bar{t} = 184$; 4) $\bar{t} = 368$; 5) $\bar{t} = 735$; 6) $\bar{t} = 1300$; $\bar{v}_0 = 0.1 \cos \bar{\lambda}\bar{y}$; $\mathrm{Rm_s} = 0.95$; $\mathrm{St} = 0.024$; $\bar{\lambda}/\bar{\delta} = 0.0235$.

$$2|\mathrm{Rm}_s - \mathrm{Rm}_k|(\mathrm{Rm}_k^2 + \mathrm{Rm}_{k0}^2)(1 + \mathrm{Rm}_k^2) + \overline{\mathrm{Ha}}^2(\mathrm{Rm}_{k0}^2 - \mathrm{Rm}_k^2) = 0, \tag{11a}$$

where the Hartmann number $\overline{\mathrm{Ha}}^2 = 2\mu_0^3 \mathrm{H}_{\mathrm{m}0}^2 \delta\sigma^2/\rho\lambda\alpha$ is independent of the field velocity v_s.

We note that the possibility of development of hydrodynamic instability in MHD throttles with constant but alternating magnetic field [4] follows from condition (11a). Actually, for $\mathrm{Rm_s} = 0$ condition (11a) gives the limit of stability for the throttle in the same way as in [4] but with multipole inductor.

4. For the numerical solution of the problem we use the method of finite differences. We introduce a space–time grid:

$$\bar{y}_i = ih; \quad \bar{t}_j = j\tau;$$

$$i = 0, 1, 2, \ldots, N; \quad h = \pi/\bar{\lambda}_1 N; \quad j = 0, 1, 2\ldots$$

and use the notation $f(\bar{y}_i, \bar{t}_j) = f_{i,j}$; $f(\bar{y}, \bar{t}_j) = f_j$. We set up the finite difference analog of Eqs. (1)-(3), (10) in the case of an explicit four-point two-layer difference scheme:

$$\bar{h}_{r;i,j+1} = \frac{\tau}{\mathrm{Rm}\, h^2}(\bar{h}_{r;i+1,j} + \bar{h}_{r;i-1,j}) + \bar{h}_{r;i,j}\left(1 - \frac{2\tau}{\mathrm{Rm}\, h^2} - \frac{\tau}{\mathrm{Rm}}\right) + \frac{\bar{h}_{a;i,j}\tau}{\mathrm{Rm}}(1 - \bar{v}_{i,j}) - \frac{\tau}{\mathrm{Rm}}\bar{v}_{i,j};$$

$$\bar{h}_{a;i,j+1} = \frac{\tau}{\mathrm{Rm}\, h^2}(\bar{h}_{a;i+1,j} + \bar{h}_{a;i-1,j}) + \bar{h}_{a;i,j}\left(1 - \frac{2\tau}{\mathrm{Rm}\, h^2} - \frac{\tau}{\mathrm{Rm}}\right) + \bar{h}_{r;i,j}\tau\, \mathrm{Rm}\,(\bar{v}_{i,j} - 1) - \frac{\tau}{\mathrm{Rm}}\bar{v}_{i,j};$$

$$\bar{v}_{i,j+1} = \bar{v}_{i,j} + \tau\, \mathrm{St}\, \bar{h}_{a;i,j} - \tau\bar{c}_j - \tau(\mathrm{sign}\, \bar{v}_{i,j})\frac{\bar{\lambda}}{\bar{\delta}}(\bar{v}_{i,j})^2; \tag{12}$$

$$\bar{c}_j = \int_0^{\pi/\bar{\lambda}_1} \frac{\bar{\lambda}_1}{\pi}\left(\mathrm{St}\, \bar{h}_{aj} - \frac{\lambda \bar{v}_j^2}{\bar{\delta}}\, \mathrm{sign}\, \bar{v}_j\right) d\bar{y}.$$

We write the boundary conditions in the form

$$\bar{h}_{a;0,j} = \bar{h}_{a;1,j}, \quad \bar{h}_{r;0,j} = \bar{h}_{r;1,j};$$
$$\bar{h}_{a;N,j} = \bar{h}_{a;N-1,j}, \quad \bar{h}_{r;N,j} = \bar{h}_{r;N-1,j}. \tag{13}$$

In the computations we took N = 50; the time step was chosen based on the consideration of the stability of the computation [5, p. 227] from the condition

$$\tau \leqslant \frac{h^2\, \mathrm{Rm}}{2}.$$

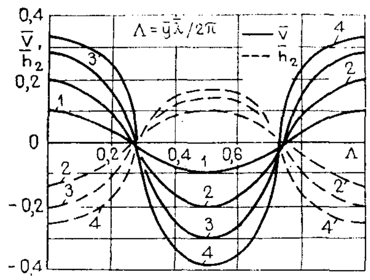


Fig. 5

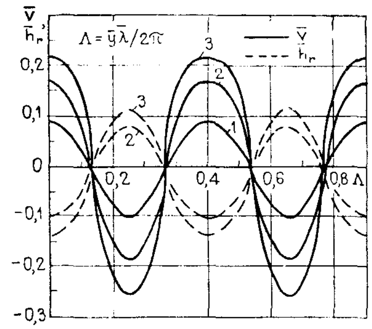


Fig. 6

Fig. 5. Development of the first supercritical motion: 1) $\bar{t} = 0$; 2) $\bar{t} = 132$; 3) $\bar{t} = 264$; 4) $\bar{t} \geq 660$; $\bar{T} = 2100$; $\bar{v}_0 = 0.1 \cos \bar{\lambda}\bar{y}$; $Rm_S = 1.59$; $St = 0.024$; $\bar{\lambda}/\bar{\delta} = 0.0235$; $\bar{\lambda}_1 = 0.342$.

Fig. 6. Development of the second supercritical motion: 1) $\bar{t} = 0$; 2) $\bar{t} = 270$; 3) $\bar{t} \geq 1280$; $\bar{T} = 4000$; $\bar{v}_0 = 0.1 \cos 2\bar{\lambda}\bar{y}$.

In evaluating the definite integral in the expression for $\bar{c}_j$ in (12) the Simpson formula was used. The solution of system (12) was obtained numerically on an M222 computer.

5. The development of velocity and pressure perturbations is shown in Figs. 2 and 3 respectively in the case of absence of friction just as in [2], $Rm_2 > Rm > Rm_1$ ($Rm = 2$; $Rm_1 = 1.86$).

The initial velocity perturbation was specified in the form $\bar{v}_0 = 0.05 \cos \bar{\lambda}\bar{y}$, i.e., in the form of the first critical motion.

The perturbation grows with time and the region of flow expands in the direction of the field so that after a certain transitional process a characteristic stationary distribution is established [2] in which it is possible to distinguish two zones: a relatively wide region of slow flow in the direction of the field and a narrow zone of reverse flow where the velocity is larger.

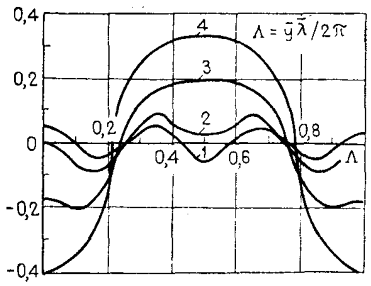
Below we consider an example in which due to the effect of the friction the supercritical motion is close in form to the corresponding critical motion. The computations were carried out for $St = 0.024$, $\bar{\lambda}/\bar{\delta} = 0.0235$, $\bar{\lambda}_1 = 0.342$ for $Rm_S = 0.95$, and $Rm_S = 1.59$.

For $Rm_S = 0.95$, $Rm_S < Rm_1$ and therefore the perturbation is damped (Fig. 4). In the case $Rm_S = 1.59$, computing $\widetilde{Ha}^2 = St(Rm_S^{-1} + Rm_S)\bar{\delta}/\bar{\lambda}$ from Fig. 1 we find that $Rm_1 = 1.32$, $Rm_2 = 1.44$, and $Rm_3 = 1.54$, i.e., in contrast to the example given above there exist first, second, and third critical motions.

In order to verify the possibility of development of the corresponding supercritical motions, the initial velocity perturbations were specified in the form $\bar{v}_0 = a \cos k\bar{\lambda}_1 \bar{y}$, where $k = 1, 2, 3$, i.e., in the form of the k-th critical motion. In the cases $k = 1$ and $k = 2$ the corresponding supercritical motions developed (Figs. 5, 6). Their form did not change when the amplitude of the initial perturbation was double compared to that taken in Figs. 5 and 6. In the case $k = 3$ the first supercritical motion developed (Fig. 7), analogous to the motion appearing for $k = 1$ (Fig. 5).

In order to elucidate the stability of the supercritical motion, we proceeded in the following way. At first, perturbation in the form of the k-th critical motion was superposed on the liquid at rest, which led to the development and establishment of supercritical motions. At a certain instant of time a perturbation in the form of some critical motion was again superposed on these motions and the evolution of the resulting motion was observed. After superposing these perturbations of different forms on the first (basic) supercritical motion, the perturbations were damped and the original supercritical motion was established. In the absence of perturbations the second supercritical motion exists in the flow for a long time (in the range of $Rm = 1.59$-2 the time is ~ 15 times greater than the time of development of the second supercritical motion); however, finally the motion goes over into the basic supercritical motion.

In analogy with [6, p. 166] the second supercritical motion can be regarded as metastable. The superposition of perturbations on this motion accelerates its transition to the basic motion (Fig. 8).



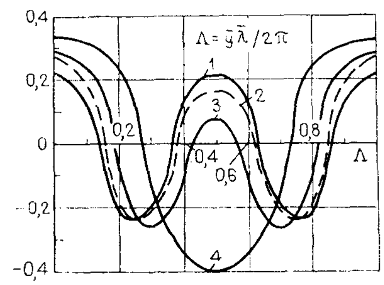


Fig. 7 Fig. 8

Fig. 7. Development of the velocity perturbation with transition of the third supercritical motion into the first: 1) $\bar t = 66$; 2) $\bar t = 462$; 3) $\bar t = 660$; 4) $\bar t = 1720$; $\bar v_0 = 0.05 \cos 3\bar\lambda\bar y$; St = 0.024; $Rm_S = 1.59$; $\bar\lambda/\bar\delta = 0.0235$; $\bar\lambda_1 = 0.342$.

Fig. 8. Transition of the second supercritical motion under the action of perturbation into the first: 1) initial motion (second supercritical); 2) perturbed motion; 4) steady-state motion (first supercritical motion); 1, 2) $\bar t = 660$; 3) $\bar t = 1850$; 4) $\bar t = 3300$; St = 0.024; $Rm_S = 1.59$; $\bar\lambda/\bar\delta = 0.0235$; $\bar\lambda_1 = 0.342$.

Conclusions

1. A numerical experiment on the development of initial perturbations in an electrically conducting liquid at rest in a traveling magnetic field has been conducted.

2. At magnetic Reynolds numbers smaller than the critical the perturbations were damped. If the magnetic Reynolds number of the homogeneous flow exceeds the critical number, then after a certain transitional process the first (basic) supercritical motion is established irrespective of the form of the initial perturbation; this motion is stable with respect to external perturbations.

3. On introducing the initial perturbation in the form of the second critical motion, the second supercritical motion develops and lasts for a long time. This motion, which finally goes over into the basic motion, is regarded as metastable.

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