NONUNIFORM MODEL OF A HELICAL DYNAMO

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Interest in geometrically simple (laminar) models in MHD dynamo theory is caused mainly by two of their characteristics: In the first place, a simpler situation permits a more rigorous and complete mathematical description, which provides for visualization and validity of the results obtained [1], and secondly, an obvious ability to excite a magnetic field together with realistic assumptions for the velocity distribution and relative smallness of the critical values of the magnetic Reynolds number make the experimental production of a uniform hydromagnetic dynamo essential [2].

The simplest motion capable of exciting a magnetic field in a uniform electrically conductive medium is uniform rigorous helical motion [3, 4]. However, an actual dynamo model always has nonuniform electric conductivity or a sharp boundary of the medium with an external insulator, which can have an appreciable effect on the self-excitation process or disrupt it completely. Taking account of the nonuniformity of the medium in the radial direction is the next step in the investigation of the helical dynamo model.

In the kinematic approximation the magnetic field is described by the equations

$$\frac{\partial \mathbf{B}}{\partial t}=\frac{1}{\sigma}\triangle\mathbf{B}+R_m\mathrm{curl}(\mathbf{v}\times\mathbf{B});\quad \mathrm{div}\,\mathbf{B}=0, \tag{1}$$

which are made dimensionless by the introduction of a characteristic length R, velocity V, and electric conductivity σ_0, and where $R_m = \mu_0\sigma_0 VR$.

The model under discussion is axisymmetrical in cylindrical coordinates r, φ, and z and consists of three coaxial cylindrical regions (Fig. 1): I) $0 < r < r_1$, II) $r_1 < r < r_2$, and III) $r_2 < r < \infty$. Rigorous helical motion with velocity components $v_r = 0$, $v_\varphi = \omega r$, and $v_z = v$ and a relative electric conductivity σ of the medium in each region are characterized by the constants ω^i, v^i, and σ^i, i = I, II, III, which in the general case are different for each of them.

By virtue of the symmetry of the model in φ and its homogeneity in z and t the solution of Eqs. (1) can be represented in the form $\mathbf{B}(r, \varphi, z, t) = (B_r(r), B_\varphi(r), B_z(r))\cdot\exp(im\varphi + ikz + pt)$, where $m = 1, 2, 3, \ldots$

In the case of rigorous motion within the limits of each region the induction equation (1) is represented in a simpler form the same as for a fixed medium upon changing to the coordinate system $r' = r$, $\varphi' = \varphi - R_m\omega t$, and $z' = z - R_m vt$, which moves rigidly along with the medium:

$$\frac{\partial \mathbf{B}}{\partial t}=\frac{1}{\sigma}\triangle'\mathbf{B}. \tag{2}$$

Here only the effective frequency of the field $p' = p + iR_m(m\omega + kv)$ varies. Thus, it is easy to derive the equations which determine the dependence of the field on r in each region:

$$LB_r-2imB_\varphi r^{-2}=0;\quad LB_\varphi+2imB_r r^{-2}=0;\quad L_0B_z=0, \tag{3}$$

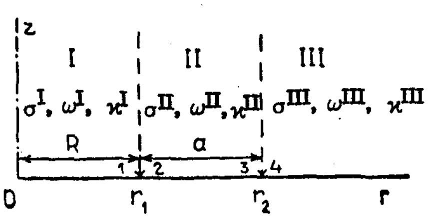

Fig. 1. Layout of the three cylindrical regions.

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where

$$L_0 = L + \frac{1}{r^2} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}}{\mathrm{d}r} - \frac{m^2}{r^2} - q^2; \quad q^2 = k^2 + \sigma p'.$$

The general solution of these equations is known in the form of the modified Bessel functions:

$$B_\pm = B_r \pm iB_\varphi = C_\pm I_{m\pm1}(qr) + D_\pm K_{m\pm1}(qr), \qquad (4)$$

$$B_z = EI_m(qr) + FK_m(qr).$$

The equation divB = 0 gives $E = (iq/2k)(C_+ + C_-)$, $F = (q/2ik)(D_+ + D_-)$. The integration constants $C_\pm$ and $D_\pm$ are determined by conditions at the region boundaries:

1) at r = 0 the field **B** is bounded,

2) as $r \to \infty$ the field $\mathbf{B} \to 0$ (absence of external sources), and

3) at $r = r_1$ and $r = r_2$ the field **B** and tangential components of the electric field

$$\mathbf{E} = \sigma^{-1}\mathrm{curl}\,\mathbf{B} - \mathrm{Rm}\,\mathbf{v}\times\mathbf{B}$$

are continuous.

The boundary conditions give a system of eighth-order homogeneous equations for the integration constants. The order of this system is reduced to fourth by direct elimination. Consistency of the system gives the dispersion relation

$$\mathrm{Det}|a_{ij}| = 0, \quad i, j = 1, 2, 3, 4, \qquad (5)$$

where

$$a_{\substack{11\\12}} = K_{m\pm1}(2)\,[P_\pm(1) + Q_\pm(2)]; \quad a_{\substack{13\\14}} = I_{m\pm1}(2)\,[P_\pm(1) - P_\pm(2)];$$

$$a_{\substack{21\\22}} = K_{m\pm1}(2)\,[u_1 - u_2 \pm \rho_1 P_\pm(1) \pm \rho_2 Q_\pm(2)];$$

$$a_{\substack{23\\24}} = I_{m\pm1}(2)\,[u_1 - u_2 \pm \rho_1 P_\pm(1) \mp \rho_2 P_\pm(2)];$$

$$a_{\substack{31\\32}} = K_{m\pm1}(3)\,[Q_\pm(4) - Q_\pm(3)]; \quad a_{\substack{33\\34}} = I_{m\pm1}(3)\,[Q_\pm(4) + P_\pm(3)];$$

$$a_{\substack{41\\42}} = K_{m\pm1}(3)\,[u_4 - u_3 \mp \rho_4 Q_\pm(4) \pm \rho_3 Q_\pm(3)];$$

$$a_{\substack{43\\44}} = I_{m\pm1}(3)\,[u_4 - u_3 \mp \rho_4 Q_\pm(4) \mp \rho_3 P_\pm(3)].$$

Here

$$P_\pm(x) = \frac{xI_m(x)}{I_{m\pm1}(x)}, \quad Q_\pm(x) = \frac{xK_m(x)}{K_{m\pm1}(x)},$$

$$u = \omega r, \quad \rho = \frac{1}{i\sigma r}.$$

The number in parentheses denotes the argument x = qr in accordance with Fig. 1, i.e., the region and boundary (for example, $(2) = q^{II}r_1$), and the subscript on u and ρ denotes the same thing.

Equation (5) is not altered upon the replacement $q^I \to -q^I$ or $q^{II} \to -q^{II}$, but $\arg q^{III} \leqslant \pi/2$ always by virtue of attenuation of the field as $r \to \infty$.

Equation (5) contains in the form of a particular case the dispersion relation of Ponomarenko's model (Eq. (8) in [3]).

The dispersion relation (5) is the transcendental equation

$$F(p, k, \mathrm{Rm}, m) = 0, \qquad (6)$$

which determines implicitly for each harmonic m of the field the dependence of the eigenvalue p on the wave number k and magnetic Reynolds number R_m. One can find the zeroes of the function F only numerically in the general case. We wrote a computer program to do this

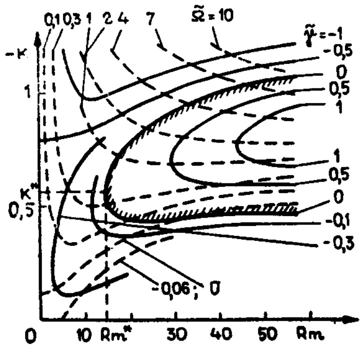

Fig. 2

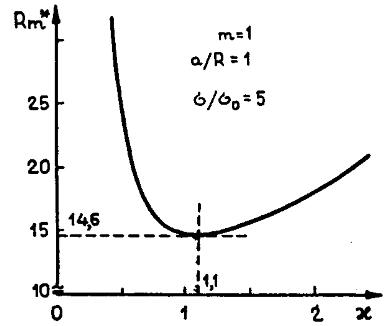

Fig. 3

Fig. 2. First eigenvalue $p(R_m, k)$ for $m = 1$, $\varkappa = 1.1$, $a = 1$, and $\sigma = 5$.

Fig. 3. Minimum of $R_m^*(\varkappa)$ in the case of a highly conductive tube.

by varying the parameters of the problem and calculating these zeroes to a specified accuracy.

Practically, the most important case is $\sigma^{III} = 0$ and $V^{II} = V^{III} = 0$, which corresponds to helical motion in a tube with a wall of finite thickness; the electric conductivity of the tube and the moving conductor are different in the general case. In order that the magnetic Reynolds number be determined in terms of the conductivity and the radius of the moving cylinder be determined in terms of the maximum velocity of the motion, one should set $\sigma^I = r_1 = \omega^{I2} + v^{I2} = 1$. The generation characteristics in Eq. (6) then still depend on three free parameters: $\varkappa = \omega^I/v^I$ is the pitch of the helical motion; $\sigma = \sigma^{II}$ is the relative electric conductivity of the tube; and $a = r_2 - r_1$ is the relative thickness of the tube walls.

One should assume the eigenvalue p in Eq. (6) to be complex, and $p = \gamma + i\Omega$ is the growth increment and frequency of the field. We will assume the wave number k to be real in order to keep the field bounded in the z coordinate. A field is excited if an eigenvalue with positive increment $\gamma \geqslant +0$ exists.

The dependence $p = p(k, R_m)$ defined by Eq. (6) is illustrated in Fig. 2 for one specific set of parameters in the form of curves γ = const and Ω = const in the (R_m, k)-plane. Here the curve $\gamma = 0$, which corresponds to neutral perturbations, is crosshatched. The far left point on this curve determines the critical magnetic Reynolds number R_m^* and consequently the critical frequency Ω^* and the critical wave number k^* of the magnetic field.

Let us turn our attention to the qualitative differences in Fig. 2 from the analogous figure for a uniform model [4, Fig. 3]. In the first place, there is no disappearance of the eigenvalue mentioned in [3] as $R_m \to 0$ and $k \to 0$, and the illustrated first branch exists in the entire (R_m, k)-plane. Nevertheless, one can maintain the numbering of the eigenvalues given in [3]. For all $R_m \neq 0$ the flow will stretch a field with small k (long wavelengths) out with itself $(-[\Omega m/(m^2 + k^2)] > 0$ is the azimuthal component of the phase velocity of magnetic field waves, and for $\omega > 0$ — of the azimuthal velocity of the medium); however, the critical and most rapidly growing modes rotate in a direction opposite to the rotation of the medium.

Both branches of the neutral curve have asymptotes as $R_m \to \infty$ of the type $k = -\text{const} \neq 0$, but the condition that the field be frozen into the moving cylinder $m + \varkappa k = -\Omega\sqrt{1 + \varkappa^2}/R_m$ as $R_m \to \infty$ is satisfied only on the upper branch.

The dependences of the critical characteristics R_m^*, k^*, and Ω^* on the parameters of the problem $\varkappa$, a, and σ given in Figs. 3-7 are obtained by the calculation of a set of neutral curves.

There is no fundamental distinction in the dependence $R_m^*(\varkappa)$ in Fig. 3 from the uniform model (see [4, Fig. 2]), which indicates the impossibility of exciting a field only by rotational $\varkappa \to 0$ or only by translational $\varkappa \to \infty$ motion even near the boundary with an insulator, in accordance with the well-known theorem of Zel'dovich [5].

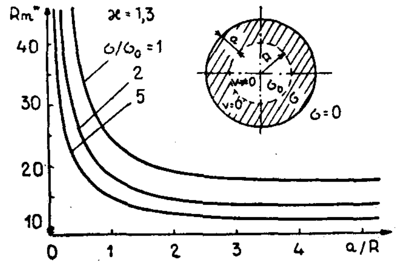

Fig. 4

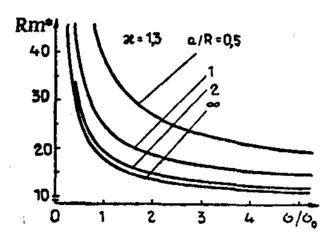

Fig. 5

Fig. 4. Dependence of the critical magnetic Reynolds number on the relative thickness of the tube wall for m = 1.

Fig. 5. Dependence of the critical wave number of the magnetic field on the wall thickness.

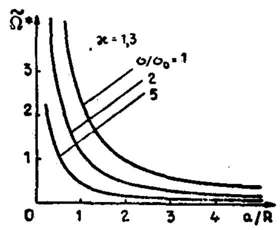

Fig. 6

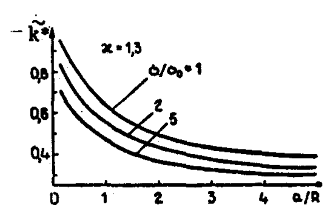

Fig. 7

Fig. 6. Dependence of the critical frequency on wall thickness.

Fig. 7. Dependence of the critical magnetic Reynolds number on the relative electric conductivity of the fixed medium.

As the thickness of the tube walls a decreases, the number R_m^* increases monotonically (Fig. 4). When $a \to 0$, generation is impossible. The thickness a restricts penetration of the current in the radial direction. As this dimension decreases, the length of the wave excited along z also decreases, i.e., k* (Fig. 5) and its frequency Ω* (Fig. 6) increase. For $a = 5$ all the critical characteristics in Figs. 4-6 already differ inappreciably from their asymptotic values as $a \to \infty$.

The dependence given in Fig. 5 shows a monotonic increase of R_m^* with a decrease in the conductivity of the external medium, which is completely natural, just as is the approach to an asymptotic value of $\sigma \to \infty$. There is in fact a very extended minimum (at $\sigma \approx 1000$) in the $R_m^*(\sigma)$ dependence, beyond which unbounded growth of R_m^* follows. The impossibility of self-excitation as $\sigma \to \infty$ is easily shown analytically.

In the case $\sigma^{II} \to \infty$ the dispersion relation (5) simplifies and reduces to the expression

$$q^{I}r_1 I_m(q^{I}r_1)\cdot I_m'(q^{I}r_1)=0. \qquad (7)$$

The root $x \equiv q^{I}r_1 = 0$ of Eq. (7) corresponds to a trivial solution. The remaining roots determined by $I_m(x) = 0$ and $I_m'(x) = 0$ correspond to damped ($\gamma < 0$) fields without oscillations ($\Omega = 0$). The root $p = 0$, which corresponds to a field maintained by a surface current flowing through an ideal conductor without damping, still satisfies the formal boundary condition $\mathbf{n} \times \mathbf{E} = 0$ in addition to the roots mentioned. Thus in this case there are no roots among those of Eq. (5) which correspond to self-excitation.

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