OSCILLATIONS OF A CONDUCTING DROP IN A MAGNETIC FIELD

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The oscillations of a liquid metal drop in a uniform constant magnetic field are considered. The oscillations of a free drop are considered, as well as those as of a drop immersed in a dielectric fluid. The characteristic equations of the oscillations are obtained. Analytical expressions are found for the frequency and damping of the fundamental axially symmetric oscillation. Results are also given for the numerical solution of the characteristic equations for the lower excited modes of a drop in a region significantly influenced by a magnetic field. The asymptotic behaviour of the spectrum in strong magnetic fields is also investigated.

When an external magnetic field is applied to a moving conductor it induces electric currents in the conductor, creates pondermotive forces and so may change the nature of its movement radically. In what follows we consider the influence of a uniform constant magnetic field on the capillary oscillations of a spherical liquid metal drop, and also find the frequency and damping rate of these oscillations. Rayleigh [1] investigated the oscillations of a drop in the absence of a magnetic field, and showed that the drop possesses a discrete set of characteristic frequencies. Each of these frequencies is degenerate, since the oscillations, which differ only in the spatial orientation of the disturbed surface, have one and the same frequency. An external magnetic field removes the equivalence of the different spatial directions, and thus also removes this degeneracy. In particular the axially-symmetrical (relative to the magnetic field) oscillations have one set of frequencies and damping times, and the asymmetrical oscillations have others. Zambran [2] using the method of perturbations calculated the damping of axially-summetric oscillations in a weak magnetic field. Here we shall consider both axially-symmetric and well as asymmetric motions by a method which allows us to find the oscillation spectrum in magnetic fields of any magnitude.

Characteristic equation. Let a uniform magnetic field B_0 exist in a vacuum in the direction of the z axis,* and let a liquid metal drop of radius R_0, density ρ, electrical conductivity σ and having a surface tension α be situated in this field. If the drop is slightly deformed ($|\delta R(\vartheta, \varphi)| \ll R_0$) from the spherical, surface tension forces return the surface of the drop ($R(\vartheta, \varphi) = R_0 + \delta R(\vartheta, \varphi)$) to its original position ($R = R_0$) by means of damped or aperiodic oscillations. Oscillations of the surface are accompanied by motion of the liquid relative to the magnetic field, and an induced electric current arises in the drop

$$\mathbf{j} = \sigma\{\mathbf{E} + [\mathbf{v}\mathbf{B}_0]\}, \tag{1}$$

which interacts with the motion of an incompressible fluid according to the equations

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}[\mathbf{j}\mathbf{B}_0], \qquad \operatorname{div}\mathbf{v} = 0. \tag{2,3}$$

When writing Eq. (2) the non-linear term $(\mathbf{v}\nabla)\mathbf{v}$ was omitted since it has been assumed that the oscillations are small. The magnetic field inside the drop $\mathbf{B}_0$ also oscillates together with the drop. However it was assumed that $\mathbf{B} = \mathbf{B}_0$ in (1) and (2), which is permissible if the radius R_0 is much less than the thickness of the skin-layer δ, corresponding to the characteristic frequency of the drop oscillations ω,

$$R_0 \ll \tilde{\delta} = \left(\frac{1}{2}\sigma\mu_0\tilde{\omega}\right)^{-1/2}. \tag{4}$$

In what follows [see (50)] it is clear that $\tilde{\omega}^2 \sim \alpha R_0^{-3}\rho^{-1}$, and so (4) is equivalent to the inequality

$$R_0 \ll \rho/(\sigma\mu_0)^2\alpha. \tag{5}$$

The inequality (5) is assumed to be fulfilled: for real metals it is violated only when the dimensions of the drop are extremely large. Even in the most unfavorable case (liquid Na) Eq. (5) means that $R_0 \ll 30$ m.

If only axially-symmetric oscillations were of interest the electric field in formula (1) could be neglected. For axially-symmetric oscillations the lines of electric current are concentric circles; no space or surface charges accumulate anywhere and $\mathbf{E} = 0$. For asymmetric oscillations on the other hand $\mathbf{E} \neq 0$. If (5) is fulfilled the electric field may be taken to be a potential field,

$$\mathbf{E} = -\nabla\Phi \tag{6}$$

and Φ may be determined from the equation

$$\operatorname{div}\mathbf{j} = 0. \tag{7}$$

Since the drop is surrounded by vacuum the lines of electric current are closed within the drop and no current intersects the surface. This means that

$$j_r|_{r=R_0} = 0, \tag{8}$$

since, because of the smallness of the oscillations under consideration, we may treat the oscillating surface as the same as the equilibrium sphere $r = R_0$.

*In what follows two coordinate systems are employed: cylindrical $(z, r_\perp, \varphi)$ with origin at the center of the drop. Components of the vector v in the cylindrical system are given by v_z, $v_{r\perp}$, v_φ, and in the spherical by *. The symbols $v_\perp$ and $\nabla_\perp$ represent the components of the vectors v and ∇ in the plane z = const. The following symbols are also used $\Delta_\perp = \Delta - (\partial^2/\partial_z^2)$ and $s = \cos\vartheta$ $(z = rs,\ r_\perp = r(1 - s^2)^{1/2}$.

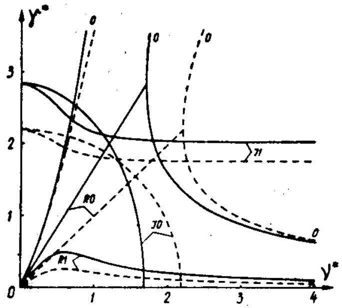

Fig. 1. Solutions of the characteristic equation; $l = 2$, m = 0, 1. In this and in all other figures the continuous lines correspond to oscillations of the free drop, broken lines to an immersed drop with $\rho_a = \rho$. The figures by each curve give the value of m. The letters R, I in front of the number indicate the real and imaginary part of the complex roots. The real roots are indicated only by the value m.

Relation (8) will be used as one of the boundary conditions for the system (1)–(3), (6), (7). A second boundary condition is Laplace's formula

$$p=\alpha\left(\frac{1}{R_1}+\frac{1}{R_2}\right), \quad (9)$$

which connects the pressure in the drop directly at its surface with its curvature $(1/R_1 + 1/R_2)$. The curvature in its turn is determined by the departure $\delta R(\vartheta, \varphi)$ of the drop surface from a sphere $R(\vartheta, \varphi) = R_0$. For $|\delta R(\vartheta, \varphi)| \ll R_0$

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{2}{R_0}-\frac{1}{R_0^2}\times\left[2+\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta}+\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right]\times\delta R(\vartheta,\varphi)=\frac{2}{R_0}-\frac{1}{R_0^2}\times\left[2+\frac{\partial}{\partial s}(1-s^2)\frac{\partial}{\partial s}+\frac{1}{1-s^2}\frac{\partial^2}{\partial\varphi^2}\right]\delta R(\vartheta,\varphi) \quad (10)$$

[see, for example, [3], formula (61.5)].

For convenience in what follows we differentiate (9) with respect to time, writing $(\partial/\partial t)\delta R = v_r|_{r = R_0}$ since the oscillations are small, and consider the resulting condition

$$\left.\frac{\partial p}{\partial t}\right|_{r=R_0}=-\frac{\alpha}{R_0^2}\left[2+\frac{\partial}{\partial s}(1-s^2)\frac{\partial}{\partial s}+\frac{1}{1-s^2}\frac{\partial^2}{\partial\varphi^2}\right]v_r\Big|_{r=R_0} \quad (11)$$

to apply to the equilibrium surface of the drop $r = R_0$.

Thus the free oscillations of the drop are described by Eqs. (1)–(3), (6), (7) with boundary conditions (8), (11). In order to determine the eigenfrequencies of the drop we look for solutions of system which are exponential functions of time

$$\mathbf{v}, p, \mathbf{j}, \Phi, \mathbf{E} \sim e^{-\gamma t}. \quad (12)$$

The frequency and damping coefficient of the oscillations are equal respectively to the imaginary and real parts of γ. Setting (12) in Eqs. (1), (2), (6) and eliminating E, we find that

$$v_z=\frac{1}{\rho\gamma}\frac{\partial}{\partial z}p, \quad (13)$$

$$j_z=-\sigma\frac{\partial}{\partial z}\Phi, \quad (14)$$

$$\mathbf{v}_\perp=\frac{1}{(\gamma-\nu)\rho}\{\nabla_\perp p+\sigma[\nabla\Phi, \mathbf{B}_0]\}, \quad (15)$$

where

$$\mathbf{j}_\perp=-\frac{\sigma}{(\gamma-\nu)}\left\{\gamma\nabla_\perp\Phi-\frac{1}{\rho}[\nabla p, \mathbf{B}_0]\right\}, \quad (16)$$

$$\nu=\sigma B_0^2\rho^{-1}. \quad (17)$$

Substituting (13)–(16) in Eqs. (2), (7) and taking the identities div $\nabla_\perp p = \Delta_\perp p$, div $[\nabla\Phi, \mathbf{B}_0] = \mathbf{B}_0$ rot $\nabla\Phi = 0$ into account, we obtain the equations for determining ρ and Φ,

$$\left[\Delta_\perp+\left(1-\frac{\nu}{\gamma}\right)\frac{\partial^2}{\partial z^2}\right]p=0, \quad (18)$$

$$\left[\Delta_\perp+\left(1-\frac{\nu}{\gamma}\right)\frac{\partial^2}{\partial z^2}\right]\Phi=0. \quad (19)$$

The quantities j and v_r contained in the boundary conditions (8), (11) may easily be expressed in terms of p and Φ with the help of (13)–(16),

$$v_r=\frac{1}{r}(r_\perp v_{r_\perp}+zv_z)=\frac{1}{r\rho(\gamma-\nu)}\left[\left(1-\frac{\nu}{\gamma}\right)z\frac{\partial p}{\partial z}+r_\perp\frac{\partial p}{\partial r_\perp}+\sigma B_0\frac{\partial\Phi}{\partial\varphi}\right], \quad (20)$$

$$j_r=\frac{1}{r}(r_\perp j_{r_\perp}+zj_z)=-\frac{\sigma\gamma}{r(\gamma-\nu)}\times\left[\left(1-\frac{\nu}{\gamma}\right)z\frac{\partial\Phi}{\partial z}+r_\perp\frac{\partial\Phi}{\partial r_\perp}-\frac{B_0}{\gamma\rho}\frac{\partial p}{\partial\varphi}\right]. \quad (21)$$

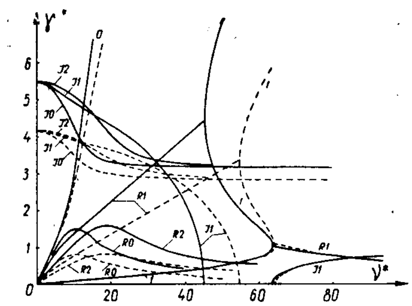

Fig. 2. Solution of the characteristic equation; $l = 3$, m = 0, 1, 2.

In the problem under consideration the angular variable φ is separable and the eigenfunctions depend upon it through the factor $e^{im\varphi}$:

$$p, \Phi \sim e^{im\varphi}, \quad m=0, \pm1, \pm2, \ldots. \quad (22)$$

If, instead of Φ, we introduce ψ such that

$$\Phi=-\frac{im}{\sigma B_0}\psi, \quad (23)$$

Eqs. (18), (19) and the boundary conditions (8), (11) acquire the following forms:

$$\left[\frac{1}{r_\perp}\frac{\partial}{\partial r_\perp}r_\perp\frac{\partial}{\partial r_\perp}-\frac{m^2}{r_\perp^2}+\left(1-\frac{\nu}{\gamma}\right)\frac{\partial^2}{\partial z^2}\right]p=0, \quad (24)$$

$$\left[\frac{1}{r_\perp}\frac{\partial}{\partial r_\perp}r_\perp\frac{\partial}{\partial r_\perp}-\frac{m^2}{r_\perp^2}+\left(1-\frac{\nu}{\gamma}\right)\frac{\partial^2}{\partial z^2}\right]\psi=0, \quad (25)$$

$$\left[r_\perp\frac{\partial}{\partial r_\perp}+\left(1-\frac{\nu}{\gamma}\right)z\frac{\partial}{\partial z}\right]\psi+\frac{\nu}{\gamma}p=0, \quad r=R_0, \quad (26)$$

$$\gamma p=\frac{\alpha}{R_0^2}\left[2+\frac{\partial}{\partial s}(1-s^2)\frac{\partial}{\partial s}-\frac{m^2}{1-s^2}\right]v_r, \quad r=R_0, \quad (27)$$

where

$$v_r=\frac{1}{(\gamma-\nu)\rho r}\times$$
$$\times\left\{\left[r_\perp\frac{\partial}{\partial r_\perp}+\left(1-\frac{\nu}{\gamma}\right)z\frac{\partial}{\partial z}\right]p+m^2\psi\right\}. \quad (28)$$

Equations (24)–(28) contain the quantity m only in the form of a square m^2, and so the eigenfrequencies of the drop do not depend on the sign of m. In other words the removal of degeneracy mentioned in the introduction is not complete and for $m \neq 0$ each frequency remains doubly degenerate. In what follows all formulas are written for the case $m \geq 0$.

A full system of bounded solutions of the equation

$$\left[\frac{1}{r_\perp}\frac{\partial}{\partial r_\perp}r_\perp\frac{\partial}{\partial r_\perp}-\frac{m^2}{r_\perp^2}+\left(1-\frac{\nu}{\gamma}\right)\frac{\partial^2}{\partial z^2}\right]u=0, \quad (29)$$

inside the drop is given by the homogeneous (with respect to r and z) polynomials

$$u^{(n)}=\sum_{i=0}^{E\left(\frac{n-m}{2}\right)}C_i^{(n)}r_\perp^{m+2i}z^{n-m-2i}, \quad n\geqslant m, \quad (30)^*$$

in which the coefficients $C_i^{(n)}$ are connected by the recurrence relation

$$C_{i+1}^{(n)}=$$
$$=-\left(1-\frac{\nu}{\gamma}\right)\frac{(n-m-2i)(n-m-2i-1)}{4(i+1)(m+i+1)}C_i^{(n)}. \quad (31)$$

The initial coefficient $C_0^{(n)}$ of formula (31) is not determined and may be chosen to be arbitrary. In order to simplify the expression which are to be investigated below we choose

$$C_0^{(n)}=\frac{(-1)^m(n+m)!}{2^m m!(n-m)!}. \quad (32)$$

On the surface of the drop $r_\perp = R_0(1-s^2)^{1/2}$; $z = R_0 s$ and

$$u^{(n)}\big|_{r=R_0}=R_0^n(1-s^2)^{m/2}\sum_{i=0}^{E\left(\frac{n-m}{2}\right)}C_i^{(n)}(1-s^2)^i s^{n-m-2i}. \quad (33)$$

In what follows it is convenient to write $u^{(n)}|_{r=R_0}$ in the form of a sum of associated Legendre functions $P_j^m(s)$:

$$u^{(n)}\big|_{r=R_0}=R_0^n\sum_{j=m}^{n}L_j^{(n)}P_j^m(s). \quad (34)$$

The behavior of the quantities

$$w^{(n)}\equiv\left[r_\perp\frac{\partial}{\partial r_\perp}+\left(1-\frac{\nu}{\gamma}\right)z\frac{\partial}{\partial z}\right]u^{(n)}=\sum_{i=0}^{E\left(\frac{n-m}{2}\right)}C_i^{(n)}\times$$
$$\times\left[m+2i+\left(1-\frac{\nu}{\gamma}\right)(n-m-2i)\right]r_\perp^{m+2i}z^{n-m-2i} \quad (35)$$

on the surface may be written down in a similar form

$$w^{(n)}\big|_{r=R_0}=R_0^n\sum_{j=m}^{n}M_j^{(n)}P_j^m(s).$$

The coefficients $L_j^{(n)}$ and $M_j^{(n)}$ are expressed in terms of $C_i^{(n)}$. It is easy to find definite formulas for $L_j^{(n)}$ and $M_j^{(n)}$ if (33) and (34) are divided by $(1-s^2)^{m/2}$, expressions $(1-s^2)^{-m/2}P_j^m(s)$ and $(1-s^2)^i$ written out in the form of polynomials and coefficients of equal powers of s equated in the resulting formulas. Thus we may ascertain, in particular, that

$$L_n^{(n)}=\frac{(-1)^m(n-m)!}{(2n-1)!!}\sum_{i=0}^{E\left(\frac{n-m}{2}\right)}(-1)^iC_i^{(n)}=$$
$$=\frac{(-1)^m(n-m)!}{(2n-1)!!}\left(\frac{\nu}{\gamma}\right)^{n/2}\left(\frac{\nu}{\gamma}-1\right)^{-\frac{m}{2}}P_n^m(\sqrt{\gamma/\nu})=$$
$$=1-\frac{(n-m)(n-m-1)}{2(2n-1)}\frac{\nu}{\gamma}+$$
$$+\frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2\cdot 4(2n-1)(2n-3)}\frac{\nu^2}{\gamma^2}-\ldots \quad (37)$$

and

$$M_n^{(n)}=\frac{(-1)^m(n-m)!}{(2n-1)!!}\sum_{i=0}^{E\left(\frac{n-m}{2}\right)}(-1)^iC_i^{(n)}\times$$
$$\times\left[m+2i+\left(1-\frac{\nu}{\gamma}\right)(n-m-2i)\right]=\left[n-\frac{\nu}{\gamma}(n-m)\right]\times$$
$$\times L_n^{(n)}+2\left(\frac{\nu}{\gamma}-1\right)\nu\frac{d}{d\nu}L_n^{(n)}. \quad (38)$$

We shall look for the solution of system (24)–(28) in the form of

$$p=\sum_n K_n p^{(n)}, \quad \psi=\sum_n K_n\psi^{(n)}, \quad (39)$$

where

$$p^{(n)}=u^{(n)}, \quad (40)$$

$$\psi^{(n)}=\sum_{k=m}^{n}\varkappa_k^{(n)}u^{(k)}. \quad (41)$$

The coefficients $\varkappa_k^{(n)}$ in (41) are chosen so that each pair of the quantities $p^{(n)}$ and $\psi^{(n)}$ separately satisfies the boundary conditon (26). Taking (34)–(36) into account this means

$$\frac{\nu}{\gamma}\sum_{j=m}^{n}L_j^{(n)}P_j^m(s)+\sum_{k=m}^{n}\varkappa_k^{(n)}\sum_{j=m}^{k}M_j^{(k)}P_j^m(s)=0. \quad (42)$$

*The symbol E(n − m/2) signifies the integral part of the number n − m/2.

Setting the coefficient of $P_n^m(s)$ equal to zero in (42), we find that

$$\varkappa_n^{(n)} = -\frac{\nu}{\gamma}\frac{L_n^{(n)}}{M_n^{(n)}}. \quad (43)$$

The remaining $\varkappa_k^{(n)}$ may easily be found by equating the other coefficients to zero.

Substituting (39) into the second boundary condition (27) and allowing for the fact that $P_j^m(s)$ satisfies the equation

$$\left[\frac{d}{ds}(1-s^2)\frac{d}{ds} - \frac{m^2}{1-s^2} + j(j+1)\right] P_j^m(s) = 0, \quad (44)$$

we have

$$\sum_n K_n \sum_{j=m}^{n} P_j^m(s) \left\{\gamma L_j^{(n)} - \frac{\alpha[2-j(j+1)]}{R_0^3\rho(\gamma-\nu)} \times \left(M_j^{(n)} + m^2 \sum_{k=j}^{n} \varkappa_k^{(n)} L_j^{(k)}\right)\right\} = 0. \quad (45)$$

Condition (45) is fulfilled for all s only if the coefficients of each $P_j^m(s)$ in this condition become zero. This gives a system of equations for determining K_n:

$$\sum_{n=j}^{\infty} \left\{\gamma L_j^{(n)} + \frac{\alpha(j+2)(j-1)}{R_0^3\rho(\gamma-\nu)} \times \left(M_j^{(n)} + m^2 \sum_{k=j}^{n} \varkappa_k^{(n)} L_j^{(k)}\right)\right\} K_n = 0,$$

$$j = m,\ m+1,\ m+2,\ \ldots,\ \infty. \quad (46)$$

It is important that each equation (46) contains not all the unkown K_n but only those for which $n \geq j$. Thus system (46) is consistent (its determinant is zero) if even one (the l-th) equation does not contain K_l, i.e., if

$$\gamma L_l^{(l)} + \frac{\alpha(l+2)(l-1)}{(\gamma-\nu)R_0^3\rho}\left(M_l^{(l)} + m^2 \varkappa_l^{(l)} L_l^{(l)}\right) = 0. \quad (47)$$

In this case all K_n with $n > l$ are equal to zero and the solutions of (39) consist of a finite sum of terms. We may write (47) in a somewhat different form if $\varkappa_l^{(l)}$ is eliminated using (43),

$$\gamma L_l^{(l)} + \frac{\alpha(l+2)(l-1)}{(\gamma-\nu)R_0^3\rho}\left(M_l^{(l)} - m^2\frac{\nu}{\gamma}\frac{(L_l^{(l)})^2}{M_l^{(l)}}\right) = 0. \quad (48)$$

The required quantities γ are the roots of the algebraic equations (48). In other words (48) are the characteristic equations for the drop oscillations.

Various types of oscillations of a free drop. Equations (48) are algebraic equations of the second or higher degree, depending on the magnitude of the numbers l and m. The equations were solved numerically for values of l from 2 to 6, and the functions $\gamma^* = \gamma\left(\frac{\rho R_0^3}{\alpha}\right)^{1/2}$ of $\nu^* = \nu\left(\frac{\rho R_0^3}{\alpha}\right)^{1/2}$ are given in Figs. 1–5 (solid curves). Equation (48) has four types of solution: 1) oscillations which are independent of the magnetic field * ; 2) oscillations which are damped in weak fields but aperiodic in strong fields; 3) oscillations which are weakly damped both in weak as well as in limitingly strong fields; and 4) purely aperiodic solutions of Eq. (48).

1. The first group comprises oscillations with m = l. For these $L_l^{(l)} = 1$, $M_l^{(l)} = m$ [see (37, 38)] and Eq. (48) assumes the form of

$$\gamma^2 + \frac{\alpha}{R_0^3\rho} l(l+2)(l-1) = 0. \quad (49)$$

Equation (49) is independent of the magnetic field. Its solution corresponds to undamped oscillations with a frequency of

$$\omega = \Im m\,\gamma = [l(l+2)(l-1)\alpha R_0^{-3}\rho^{-1}]^{1/2}. \quad (50)$$

Using formulas (13)–(16), (30), (39), and (40) it can be established that the motion of the fluid in these oscillations is plane (occuring in the planes z = const) and independent of z. Although the magnetic lines of force interesect the fluid an electric field is formed in the drop such that there is no current (j = 0) and the magnetic field exerts no influence on the motion of the fluid.

2. The second group comprises the axially-symmetric oscillations (m = 0) with even l. For l = 2 (fundamental oscillation) Eq. (48) is a quadratic,

$$\gamma^2 - \frac{\nu}{3}\gamma + \frac{8\alpha}{R_0^3\rho} = 0, \quad (51)$$

and has the solution

$$\gamma = \frac{\nu}{6} \pm \left(\frac{\nu^2}{36} - \frac{8\alpha}{R_0^3\rho}\right)^{1/2}. \quad (52)$$

In weak magnetic fields ($\nu^2 < 288\alpha R_0^{-3}\rho^{-1}$) the fundamental axially-asymmetric oscillations are damped with $\Re e\gamma = \nu/6$ and $\omega = \Im m\gamma = (8\alpha R_0^{-3}\rho^{-1} - \nu^2/36)^{1/2}$. For $\nu^2 > 288\alpha R_0^{-3}\rho^{-1}$ the solution divides into two aperiodic branches (see Fig. 1): a short lived (for $\nu \to \infty$ $\gamma \approx \nu/3$) and a long lived (for $\nu \to \infty$ $\gamma \approx 24\alpha R_0^{-3}\rho^{-1}\nu^{-1}$). The cases for $l > 2$ lead to equations of higher order $(1/2)l + 1$, however the character of the solution remains the same. In weak fields the oscillations are weakly damped and in strong fields aperiodic. In the latter case the longest lived of these relaxes according to the law

$$\gamma \approx l(l+2)(l^2-1)\alpha R_0^{-3}\rho^{-1}\nu^{-1}. \quad (53)$$

3. Axially symmetric (m = 0) oscillations with odd l and all oscillations with $0 < m < l$ comprise the third group. In both limiting cases (in weak and strong fields) they are weakly damped. In weak fields the oscillations occur with frequency (50) and damping *

$$\Re e\,\gamma = \frac{\nu(l^2-m^2)(l-1)}{2l^2(2l-1)}. \quad (54)$$

In strong fields ($\nu^2 \gg \alpha R_0^{-3}\rho^{-1}$) the oscillations have an asymptotic frequency independent of the magnetic

*To simplify the drawings these solutions are not given in Figs. 1–5.

*Zambran [2] derived formula (54) for m = 0, using the method of perturbations. It refers to the second type of oscillation considered above.

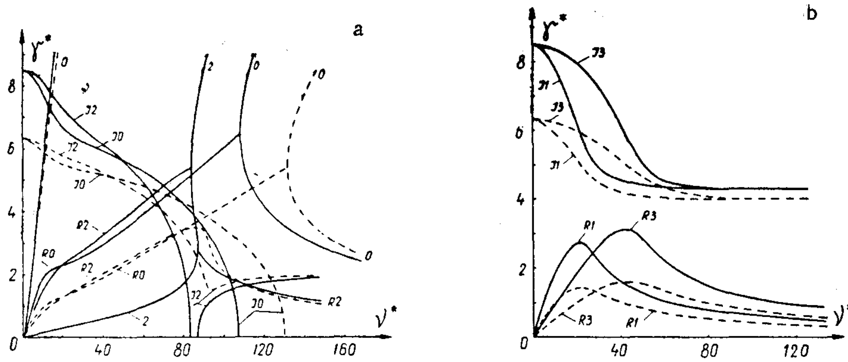

Fig. 3. Solutions of the characteristic equation: a) $l = 4$, m = 0, 2.
b) $l = 4$, m = 1, 3.

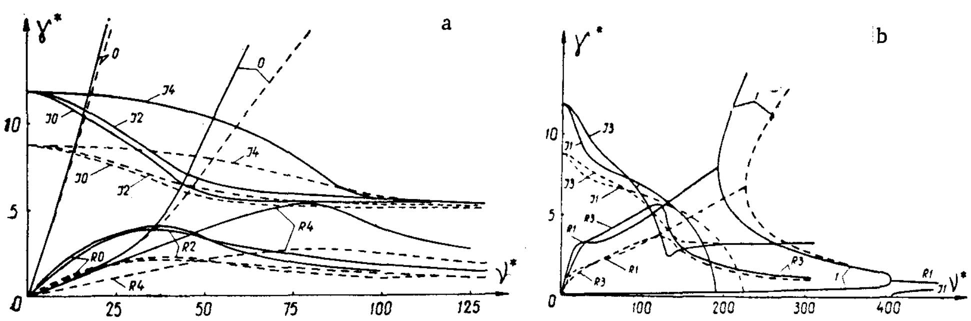

Fig. 4. Solutions of the characteristic equation: a) $l = 5$, m = 0, 2, 4;
b) $l = 5$, m = 1, 3.

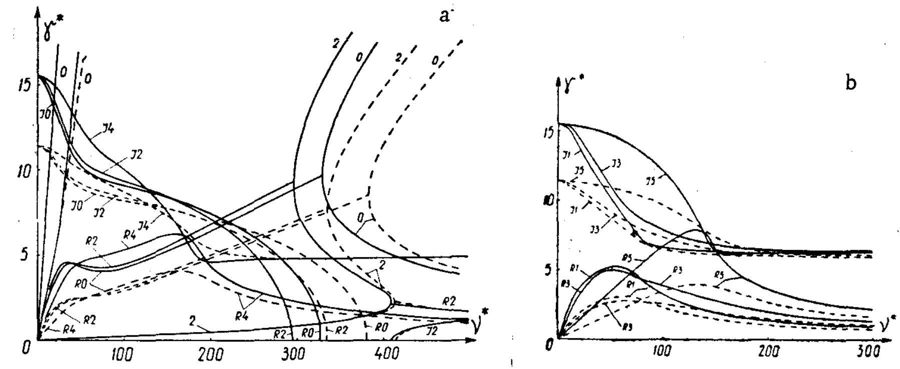

Fig. 5. Solutions of the characteristic equation: a) $l = 6$, m = 0, 2, 4;
b) $l = 6$, m = 1, 3, 5.

field, which however has different expressions for even and odd differences $l - m$. For even $l - m$

$$\omega \approx \left(\frac{\alpha}{R_0^3 \rho} \frac{(l+2)(l-1)m^2}{l^2 - m^2 + l} \right)^{1/2}, \tag{55}$$

$$\Re e\, \gamma \approx \frac{\alpha (l+2)(l-1)(l^2-m^2)[(l+1)^2 - m^2][3l(l+1) - 2m^2]}{6 R_0^3 \rho \nu [l(l+1) - m^2]}. \tag{56}$$

For odd $l - m$

$$\omega \approx [(l+2)(l-1)\alpha R_0^{-3} \rho^{-1}]^{1/2}, \tag{57}$$

$$\Re e\, \gamma \approx \frac{\alpha}{6} \frac{(l+2)(l-1)}{R_0^3 \rho \nu} (l^2 + 2m^2 + l - 2). \tag{58}$$

The different form of formulas (55), and (57), (58) is due to the radically different nature of motion of the liquid. For odd $l - m$ the light damping (58) is explained by the fact that the liquid oscillations occur basically along the magnetic field. The evenness of $l - m$ indicates that the liquid motion is symmetric relative to the plane $z = 0$. Owing to incompressibility such motion cannot occur along the magnetic field only. Thus velocity intersects the magnetic field lines. In a very strong magnetic field the liquid acquires a path of motion such that the electric field which arises cancels the induced currents. It should be noted that electric fields cannot arise in the case of axially-symmetric motion, and so all axially-symmetric oscillations with even l in strong fields are aperiodic.

The distinction between even and odd differences $l - m$ appears also in the location of the region of applicability of formulas (55)-(58), which for odd values of $l - m$ commences at lower ν than for even values. In addition to this, in the transition region between $\nu^2 \ll \alpha R_0^{-3} \rho^{-1}$ and $\nu^2 \gg \alpha R_0^{-3} \rho^{-1}$ the damping of oscillations with even values of $l - m$ is considerably larger than of oscillations with odd values. The part of the oscillations with even $l - m$ in this region becomes aperiodic. At the same time for oscillations with $l = 2$ and $m = 1$ the ratio $\Re e\gamma / \Im m\gamma$ is not large even in the transition region and does not exceed 0.212.

4. The solutions of Eqs. (48), forming a pair of pure imaginary roots for $\nu = 0$ was discussed above. With the exception of ($l = m$) and ($l = 2, m = 0$) when (48) is quadratic, its order is greater than the second in the remaining cases and it also has other roots. For $\nu = 0$ they are all equal to zero and represent different vorticial flow patterns inside the drop, totally unconnected with the motion of the surface and in an ideal liquid are undamped. The magnetic field brakes these flows and connects them to the motion of the surface. The majority of these roots remain aperiodic as ν increases. Interesting phenomena may be seen in Figs. 2–5 when, with the increase of the magnetic field, the Rayleigh branch divides into two real roots the smaller of which later fuses with one of the aperiodic roots with the formation of a pair of complex roots, following formulas (55), (56) in strong fields. The great number of roots which are considerably aperiodic does not allow us to represent them all in Figs. 2–5. All these roots are represented only in Fig. 1 ($l = 2$). All aperiodic roots with $m = 0$ are drawn in the remaining figures. For the remaining values of m only those real roots are given which have branching points in common with the Rayleigh branch.

Oscillations of an immersed drop. A more complicated problem may be solved by the method outlined above, that of the oscillations of an immersed drop, i.e., a liquid metal drop situated in an infinite space filled with another ideal non-conducting fluid with density ρ_a. Equations (1)–(3), (6), (7) or (13)–(19), (24), (25) and their solutions (40), (41), determining the motion inside the drop remain the same as in the first section. The first boundary condition (8) or, what is the same thing (26), still remains valid also. Only the second boundary condition (9), in which the pressure p_a of the surrounding fluid appears, changes,

$$p - p_a = \alpha \left(\frac{1}{R_1} + \frac{1}{R_2} \right). \tag{59}$$

In addition to this we must add the equation [the analog of Eq. (18) with $\nu = 0$]

$$\Delta p_a = 0, \tag{60}$$

which determines the external pressure, and the relation [the analog of (20)]

$$v_{ra} = \frac{1}{\gamma \rho_a} \frac{\partial}{\partial r} p_a, \tag{61}$$

connecting p_a with the radial velocity v_{ra} in the region $r \geq R_0$.

Outside the drop the functions

$$p_{a,q} = r^{-q-1} P_q^m(s), \tag{62}$$

$$v_{ra,q} = -\frac{q+1}{\gamma \rho_a} r^{-q-2} P_q^m(s), \tag{63}$$

form a complete system of bounded solutions of Eqs. (60), (61) [taking (22) into account], while the following distributions $v_{ra} = v_r$ and p_a on the surface of the drop [see (28), (34)–(36), (40), (41)]:

$$v_r^{(n)}\Big|_{r=R_0} = \frac{R_0^{n-1}}{(\gamma - \nu)\rho} \sum_{j=m}^{n} \times \left(M_j^{(n)} + m^2 \sum_{k=j}^{n} \varkappa_k^{(n)} L_j^{(k)} \right) P_j^m(s), \tag{64}$$

$$p_a^{(n)}\Big|_{r=R_0} = -\frac{R_0^n \rho_a}{\left(1 - \frac{\nu}{\gamma}\right)\rho} \sum_{j=m}^{n} \frac{1}{j+1} \times \left(M_j^{(n)} + m^2 \sum_{k=j}^{n} \varkappa_k^{(n)} L_j^{(k)} \right) P_j^m(s), \tag{65}$$

correspond to the n-th solution of (40) of the internal problem.

Carrying out all the operations on the boundary condition (59) which were performed on (9) in deriving (48), and taking (65) into account, we find the characteristic equation

$$\gamma L_l^{(l)} + (\gamma - \nu)^{-1} \left(\frac{\gamma^2 \rho_a}{(l+1)\rho} + \frac{\alpha (l+2)(l-1)}{R_0^3 \rho} \right) \times \left(M_l^{(l)} - m^2 \frac{\nu}{\gamma} \frac{(L_l^{(l)})^2}{M_l^{(l)}} \right) = 0, \tag{66}$$

which coincides with (48) for $\rho_a = 0$.

The oscillations of an immersed drop are characterised by roughly the same behavior as those of a free drop. In particular for $l = m$ we have undamped oscillations of frequency

$$\omega=\left(\frac{\alpha(l+2)(l^2-1)l}{[(l+1)\rho+l\rho_a]R_0^3}\right)^{1/2}. \qquad (67)$$

In weak fields ($\nu^2 \ll \alpha R_0^{-3}\rho^{-1}$) oscillations with l = m also have the frequency (67), only they are damped according to the law

$$\Re e\,\gamma=\frac{\nu(l^2-m^2)(l-1)}{2l^2(2l-1)}\left(1+\frac{l\rho_a}{(l+1)\rho}\right)^{-1}. \qquad (68)$$

For l = 2, m = 0 Eq. (66) is a quadratic just as was (48) and has the solution

$$\gamma=\frac{\nu\rho}{2(3\rho+2\rho_a)}\pm$$

$$\pm\left(\frac{\nu^2\rho^2}{4(3\rho+2\rho_a)^2}-\frac{8\alpha}{R_0^3(\rho+{}^2/_3\rho_a)}\right)^{1/2}. \qquad (69)$$

As ν increases axially-symmetric oscillations with even l become aperiodic and in strong fields ($\nu^2 \gg \gg \alpha R_0^{-3}\rho^{-1}$) the longest lived perturbation relaxes according to the same law (53) as for the free drop.

Just as above, for odd l with m = 0, and for all l with 0 < m < l in strong fields weakly damped oscillations exist having

$$\omega=\left(\frac{\alpha(l+2)(l^2-1)}{R_0^3[\rho_a+(l+1)\rho]}\right)^{1/2}, \qquad (70)$$

$$\Re e\,\gamma=\frac{\alpha(l^2+2m^2+l-2)}{6R_0^3\rho\nu[1+\rho_a/(l+1)\rho]^2}, \qquad (71)$$

for odd l − m, and

$$\omega=\left(\frac{\alpha(l+2)(l^2-1)m^2}{R_0^3[m^2\rho_a+(l+1)(l^2-m^2+l)\rho]}\right)^{1/2}, \qquad (72)$$

$$\Re e\,\gamma=\left[\alpha(l+2)(l^2-1)(l+1)(l^2-m^2)\times\right.$$
$$\times[(l+1)^2-m^2][3l(l+1)-2m^2]\rho\Big]\times$$
$$\times\left[6R_0^3\nu[\rho_a m^2+(l+1)(l^2+l-m^2)\rho]^2\right]^{-1} \qquad (73)$$

for even l − m.

Equations (66) have been solved numerically for the case when the densities of the drop and the surrounding liquid are equal ($\rho = \rho_a$). Curves for $\gamma(\nu)$ with $\rho = \rho_a$ and l = 2, 3, 4, 5, 6 are given in Figs. 1–5 by the broken lines.

As has already been mentioned the series of roots of Eq. (66) for γ = 0 refers to motions unconnected with displacement of the surface. In weak fields the resulting surface displacements are small and so the presence of an external fluid alters these roots by insignificantly, and the corresponding curves $\gamma(\nu)$ coincide initially. For l = 6 the differences of γ_4 are so limit minute that they cannot be represented separately within the limits of Fig. 5.

It is clear from the figures [see also (70–73)], that the presence of an external liquid exerts considerably less influence on weakly damped oscillations in strong magnetic fields than in weak fields. This is explained by the fact that without the magnetic field roughly the same volumes (~$1/l$ part of the volume of the drop) of internal and external liquid participate actively in the oscillations. The magnetic field does not change the quantity of oscillating external fluid but sets the whole volume of the drop into motion. As a result the external fluid comprises only a small part of the whole moving mass.

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