SOME NOTES ON THE THEORY OF CONDUCTING FLUID JETS

A. B. Tsinober and E. V. Shcherbinin

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1. A large number of solutions are known at present for the problem of a plane jet of conducting fluid flowing into an immersed space with a transverse magnetic field having various configurations [1–8]. For the particular case of a uniform magnetic field Peskin [3] obtained a system of equations for the stream function when expanded in a series in the small parameter $mx^{4/3}$, the Stewart number, and solved the equations numerically for the first approximation (the zeroth approximation corresponds to a solution with no magnetic field). Subsequently Smith and Cambell [7] found an analytic form for this solution. We shall show that any approximation in $mx^{4/3}$ may be obtained as a direct consequence of the results of paper [6]. Actually it follows from this paper that the solution for the dimensionless longitudinal component of velocity $\bar{u}$ can be represented in the form

$$\bar{u}=\left(1-\frac{3}{4}\bar{N}\right)\left[1-\operatorname{th}^2\left(\alpha\eta\sqrt{1-\frac{3}{4}\bar{N}}\right)\right], \qquad (1)$$

where $\bar{N} = mx^{4/3}$, and the expressions for m and α may be found from papers [3, 7] quoted above. Expanding this solution in a series in $\bar{N}$ we may obtain any of the approximate solutions. Thus the zeroth approximation corresponds to the expression

$$\bar{u}\Big|_{\bar{N}=0}=1-\operatorname{th}^2\alpha\eta,$$

and the first approximation to

$$f_1'(\eta)=\frac{\partial\bar{u}}{\partial\bar{N}}\Big|_{\bar{N}=0}=\frac{3}{4}\left[1-\operatorname{th}^2\alpha\eta\right]\left[\alpha\eta\operatorname{th}\alpha\eta-1\right]$$

(this corresponds exactly with the first approximation obtained in [7]) etc. Thus it may easily be established that Peskin's series converges to expression (1) for all values $\bar{N} < 4/3$.

2. In paper [8] an integral method was applied in solving jet problems, while for obtaining actual solutions use was made of the velocity profile resulting from the corresponding hydrodynamic problem. This enabled the volume of computing work to be shortened considerably, but the results obtained in this way, and in particular the total blurring of the jet at a finite distance from the source in the presence of a uniform magnetic field, have met with objections on the part of some workers. We shall show that the use of any other profile obeying the appropriate boundary conditions leads to the same results. In fact if the velocity u can be represented in the form $u = u_m f(\eta)$, where $\eta = y/\delta$, then the system of equations for determining the maximum velocity $u_m(x)$ and the jet "width" $\delta(x)$, are, for the example of a plane jet, as follows:

$$\frac{d}{dx}u_m^2\delta=-\alpha N u_m\delta,$$

$$\frac{du_m}{dx}=-\left(\frac{\beta}{\delta^2}+N\right),$$

where

$$\beta=-f''(0),\quad \alpha=\int_{-\infty}^{\infty}f(\eta)d\eta\Big/\int_{-\infty}^{\infty}f^2(\eta)d\eta,\quad N=\frac{\sigma B^2}{\rho}.$$

The solution of this system of equations leads to the following dependence of δ on x:

$$\int_0^{\delta}\frac{\delta^{1/2}d\delta}{\left[N(2-\alpha)\delta^2+2\beta\right]^{\frac{8-3\alpha}{4(2-\alpha)}}}=Cx.$$

The convergence conditions for this integral when $\delta\to\infty$ determine the cases when the jet is totally blurred at a finite distance from the source. Allowing for the fact that $\beta > 0$ we may write for $N > 0$ and $a > 0$

$$\int_a^{\infty}\frac{\delta^{1/2}d\delta}{\left[N(2-\alpha)\delta^2+2\beta\right]^{\frac{8-3\alpha}{4(2-\alpha)}}}\leqslant\int_a^{\infty}\frac{\delta^{1/2}d\delta}{\left[N(2-\alpha)\delta^2\right]^{\frac{8-3\alpha}{4(2-\alpha)}}},$$

from which it follows that the integral converges on condition that $\alpha < 2$. This condition is fulfilled for a wide class of velocity profiles. Thus for $f(\eta) = 1 - \eta^2$ $\alpha = 5/4$, for $f(\eta) = (1 - \eta^2)^2$ $\alpha = 448/353$, etc.*

Other methods of solution [2, 6] lead to similar results. In [6] it is proposed that this phenomenon is associated with the fact that at sufficiently large distances from the source the MHD boundary layer equations no longer give a valid description of the jet. We shall show by a very simple example that by using the full equations of motion we are also enabled to obtain a solution for that cross section (we shall call it the critical cross section) up to which the boundary layer equations give solutions.

3. Since the jet is blurred much more quickly when there is a transverse magnetic field present than when there is no field, we may neglect inertial terms at a sufficiently large distance from the source and write the equation of motion in the form

$$0=\nu\left(\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial x^2}\right)-Nu. \qquad (2)$$

In order to estimate the behavior of the jet close to the critical cross section we shall apply a method similar to the integral method, and so we set $u = u_m f(y/\delta(x))$. Setting this expression in Eq. (2) and subsequently integrating with respect to y from $-\infty$ to $+\infty$ gives

$$\frac{d^2}{dx^2}u_m\delta-\frac{N}{\nu}u_m\delta=0. \qquad (3)$$

To obtain the second equation in u_m and δ we consider (2) on the axis of the jet, i.e., for y = 0. Assuming that $f(0) = 1$, and $f''(0) = -\beta$ we obtain

$$u_m''-\frac{N}{\nu}u_m-\beta\frac{u_m}{\delta^2}=0. \qquad (4)$$

We shall look for a solution of systems (3) and (4) for large x. Then it follows from Eq. (3) that

$$u_m\delta=Ce^{\sqrt{N/\nu}\cdot x}. \qquad (5)$$

Moreover from Eq. (3) and (4) we can get

$$t''+2\sqrt{\frac{N}{\nu}}t'=\beta t^3, \qquad (6)$$

where $t = (1/\delta(x))$.

For large x we may set the right hand side of Eq. (6) equal to zero in the first approximation. The solution so obtained may be used to construct the second approximation by setting it in the right hand side of Eq. (6). Finally we have in the second approximation:

*In this connection it must be admitted that the note in article [8] on the use of a polynomial representation for the velocity is incorrect.

$$\delta = \frac{A}{e^{-2\sqrt{N/\nu}\cdot x} + \frac{\beta\nu}{36N} e^{-6\sqrt{N/\nu}\cdot x}},$$

$$u_m = B\left(e^{-\sqrt{N/\nu}\cdot x} + \frac{\beta\nu}{36N} e^{-5\sqrt{N/\nu}\cdot x}\right).$$

The constants A and B may be found by combining the solution so obtained with the solution of the boundary layer [6].

Thus, when allowance is made for the additional terms in the equation of motion, we are enabled to extend the solution beyond the critical cross section.

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