FORM OF SURFACE INSTABILITY OF A FERROMAGNETIC FLUID

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The formation of free-surface instability in a ferromagnetic fluid in an external magnetic field is studied. The calculation is similar to that made earlier by V. M. Zaitsev and M. I. Shliomis, except that, instead of one plane wave, the sum of three plane waves turned 120° relative to one another about the vertical axis is used as the initial solution of the linear problem. Because of this, nonlinear effects appear on a lower order, and the instability is always "hard" and results in the experimentally established "hexagonal" pattern with well-defined peaks.

Theoretical and experimental studies [1] of the horizontal free surface of a ferromagnetic fluid in a vertical magnetic field show that when the critical field H_k is reached the magnetostatic forces disrupt the surface stability. On the basis of a nonlinear calculation in [2] it is asserted that the conditions of the onset of instability vary greatly according to the permeability μ of the fluid. If $\mu < 3.54\ldots$, the instability is "soft," i.e., it increases gradually with an increase in H: when $H < H_k$, the only equilibrium surface is a plane, but when $H > H_k$, it is a wavy surface with the standing-wave amplitude $a \sim (H - H_k)^{1/2}$. But if $\mu > 3.54\ldots$, the instability is "hard," i.e., when H_k is reached a discontinuity produces a wave of finite amplitude, which continues to exist even at $H < H_k$. This has been established for a surface perturbation in the form of a single (distorted by higher harmonics) cosine wave. But the problem in question is a degenerate one, so that the solution of the linear problem is the superposition of any number of waves of the same length but of any amplitude, phase, and direction. The result of a nonlinear consideration depends upon the type of superposition used, so that it is necessary to find the set of initial waves that gives the "hardest" instability.

In the present calculation, the perturbation is represented as the sum of three identical cosine waves turned 120° relative to one another about the vertical axis. This perturbation corresponds to the experimentally observed "hexagonal" pattern [1] and is an elementary symmetric perturbation, the square of which is not orthogonal to the perturbation itself. The calculation is essentially similar to that in [2], but because of this nonorthogonality, the nonlinearity manifests itself on a lower order and always results in "hard" instability.

Let us assume that the upper half-space (−) is a vacuum and that the lower (+) is a ferromagnetic fluid with constant permeability $\mu^+ = \mu$ and density β and is in a gravity field g and an external magnetic field $H_0 = H_z^-|_{z=\infty} = H_z^+|_{z=\infty}$. The interface $z = \zeta(x, y)$ has the tension α and its mechanical equilibrium is determined by the condition*

$$\tfrac{1}{2}\mu_0\{\mu(H_n^2 - \mathbf{H}_t^2)\} + \rho g\zeta + \alpha(R_1^{-1} + R_2^{-1}) = \text{const.} \quad (1)$$

If we measure length in the units $(\alpha/\rho g)^{1/2}$ and assume that the curvature $(R_1^{-1} + R_2^{-1})$ equals $-\Delta\zeta$, which provides entirely satisfactory accuracy in this case, (1) is simplified and takes the form

$$\zeta - \Delta\zeta + \tfrac{1}{2}\mu_0(\alpha\rho g)^{-1/2}\{\mu(H_n^2 - \mathbf{H}_t^2)\} = \text{const.} \quad (2)$$

The normal H_n and tangential $\mathbf{H}_t$ components of the magnetic field at the interface obey the boundary conditions

$$\{\mathbf{H}_t\} = 0, \quad \{\mu H_n\} = 0. \quad (3)$$

It will be convenient to use the notations

$$C = 2[\cos x + \cos \tfrac{1}{2}(x + \sqrt{3}y) + \cos \tfrac{1}{2}(x - \sqrt{3}y)], \quad (4)$$

$$\mathbf{S} = \operatorname{grad} C,$$

which satisfy the relations

$$C^2 = 6 + 2C + \text{higher harmonics}$$
$$\mathbf{S}^2 = 6 + C + \text{higher harmonics}$$
$$C\mathbf{S} = \mathbf{S} + \text{higher harmonics} \quad (5)$$

The solution of Eqs. (2) and (3) is sought as

$$\zeta = aC + \text{higher harmonics} \quad (6)$$

$$H_z^- = \mu^{-1}H_0(\mu + b^- e^{-z}C) + \text{higher harmonics}$$
$$H_z^+ = \mu^{-1}H_0(1 + b^+ e^{z}C) + \text{higher harmonics}$$
$$H_{x,y}^\pm = \pm\mu^{-1}H_0 b^\pm e^{\pm z}S_{x,y} + \text{higher harmonics} \quad (7)$$

In (7) it is taken into account that $\mathbf{H}^\pm$ obeys the equations $\Delta\mathbf{H} = 0$ and div H = 0. The form of the fundamental harmonics corresponds to the solution of the linearized problem, except that the coefficients a and $b^\pm$ are determined from nonlinear equations (2) and (3). Wavelength variation is not considered as a higher-order effect.

In investigating the nature of the instability, it is sufficient to limit ourselves to small a. When (7) is substituted into the equations, therefore, $\exp(\pm\zeta)$ is

*$\{A\}$ is the discontinuity of A at the interface: $\{A\} = A^+|_{z=\zeta} - A^-|_{z=\zeta}$

expanded into a series, only the linear and quadratic terms in a and $b^{\pm}$ are retained in the equations, and the products of S and C are transformed by means of (5). Although the higher harmonics in (6) and (7) have the order a^2, they are omitted, since their inverse effect on the fundamental harmonic is on the order of a^3.

Conditions (3) are used in the form

$$\{H_{x,y}\} = -\{H_z\} \operatorname{grad}_{x,y} \zeta, \quad \{\mu H_z\} = \{\mu \mathbf{H}\} \operatorname{grad} \zeta, \quad (8)$$

and, along with (6) and (7), they allow us to express $b^{\pm}$ in terms of a:

$$b^{+} = \frac{\mu-1}{\mu+1} a - \frac{2(\mu-1)}{(\mu+1)^2} a^2 + O(a^3),$$

$$b^{-} = \mu \frac{\mu-1}{\mu+1} a + \frac{2\mu^2(\mu-1)}{(\mu+1)^2} a^2 + O(a^3). \quad (9)$$

Substitution of (6), (7), and (9) into (2) results in the equilibrium condition

$$\left(\frac{H_0^2 - H_k^2}{H_k^2} + \frac{3(\mu-1)}{2(\mu+1)} a \right) a = O(a^3), \quad (10)$$

$$H_k^2 = 2\mu(\mu+1)(\mu-1)^{-2} \mu_0^{-1} (\rho g \alpha)^{1/2}. \quad (11)$$

Thus, at small a both $a = 0$ and

$$a = \frac{2(\mu+1)}{3(\mu-1)} \frac{H_k^2 - H_0^2}{H_k^2} \quad (12)$$

are equilibrium values.

Since when $H_0 < H_k$ the plane surface $a = 0$ is stable, Eq. (12) corresponds to unstable equilibrium. When $H_0 > H_k$, $a = 0$ is unstable and Eq. (12) is stable. In both cases, a deviation from unstable equilibrium in the direction of increasing a can lead to either stable equilibrium with finite a (not determinable from (10), in view of the assumption $a \ll 1$) or disturbance of the surface. The resulting situation is characteristic of "hard" instability.

Proportionality between a and $H_0^2 - H_k^2$ is characteristic of a perturbation in the form of the symmetric sum of three waves and is impossible with only one or two waves. This evidently explains the "hexagonal" pattern observed in [1]. The preference of positive a means that the maxima on the surface are sharper and better-defined than the minima (the height of the maxima of the function is twice as great as the depth of the minima). This phenomenon is clearly expressed experimentally. The lack of expression of the "hard" nature of the instability is obviously due to the low value of the threshold discontinuity.

The magnetic nonlinearity of the fluid $\mu \neq$ const was not taken into account in the calculation. But this would hardly change the "hard" nature of the instability, since it follows from the very fact that (10) has a quadratic term.

REFERENCES

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