STABILITY OF HARTMANN FLOW WITH RESPECT TO TWO-DIMENSIONAL PERTURBATIONS OF FINITE AMPLITUDE

K. B. Pavlov and S. L. Simkhovich

UDC 538.4

This paper considers the hydrodynamic stability of Hartmann flow with respect to plane two-dimensional perturbations of finite amplitude. On the basis of calculations made for values of the Hartmann number equal to 0, 1, and 2, plots are made of the dependences of the values of the critical Reynolds number, at which loss of stability of laminar flow conditions sets in, on the amplitude of the perturbation.

The theory of the hydrodynamic stability of laminar flows with respect to infinitely small perturbations (the linear theory [1, 2]), in spite of the considerable advances made, leads, in many cases, to a considerable divergence from the experimental data. For example, with the gradient flow of a viscous Newtonian liquid between parallel surfaces, in accordance with the linear theory, loss of the stability of laminar flow conditions should set in at a value of the critical Reynolds number $Re_{cr} \approx 5000$, while the experimentally observed value is $Re_{cr} \approx 1000$. This is one of the main stimulating factors for the creation of the theory of a finite amplitude (the nonlinear theory, see for example [3-6]), the conclusions of which are in incomparably better agreement with the experimental data.

The divergences between the linear flow and experiment are found to be particularly significant in the investigation of the stability of the plane gradient flow of a conducting Newtonian liquid between parallel surfaces, in a transverse magnetic field (Hartmann flow). In particular, at large values of the Hartmann number, Ha, the linear theory predicts loss of stability at $Re_{cr} \approx 50{,}000$ Ha, while it follows from experiment that the value is $Re_{cr} \approx 225$ Ha. The fact of the considerable divergence of the conclusions of the linear theory of hydrodynamic stability from the experimental data renders important a consideration of the stability of magnetohydrodynamic Hartmann flow, within the framework of the nonlinear theory. A detailed examination is carried out in the present article, in which the question of the stability of Hartmann flow with respect to plane, two-dimensional perturbations of finite amplitude is studied at small values of Ha.

It is well known that steady-state Hartmann flow, taking place in the direction of the x axis, is characterized by the following distribution of the velocity $U_{0x}(y)$ and the induced magnetic field $B_{0x}(y)$ [7]:

$$U_{0x}(y) = \frac{\operatorname{ch} \mathrm{Ha} - \operatorname{ch} \mathrm{Ha}\, y}{\operatorname{ch} \mathrm{Ha} - 1}; \quad B_{0x}(y) = \frac{\mathrm{Re}_m}{\mathrm{Ha}} \frac{\operatorname{sh} \mathrm{Ha}\, y - y \cdot \operatorname{sh} \mathrm{Ha}}{\operatorname{ch} \mathrm{Ha} - 1}. \tag{1}$$

Here $-1 \leq y \leq 1$; Re_m is the magnetic Reynolds number, characterizing the conducting properties of the liquid; the remaining definitions are the conventional ones. In writing the dimensionless expressions (1), as the characteristic values of the length, the velocity, and the induction of the magnetic field, we take, respectively, the half-width of the channel, the maximal velocity at the center of the channel, and the induction of a constant external field, perpendicular to the walls of the channel and directed along the y axis.

In a consideration of the stability of the flow with respect to perturbations of small but finite amplitude, we must, in the first place, take account of the effect of the perturbations of the main flow which, in this connection, cannot be described in terms of distributions (1). To such a deformed flow which, in what

Translated from Magnitnaya Gidrodinamika, No. 1, pp. 58-64, January-March, 1972. Original article submitted August 9, 1971.

follows, we shall call the mean flow and designate by U^c, B^c, let there be applied small, but finite, perturbations of the velocity $u'\{u'_x, u'_y\}$ and of the magnetic induction $b'\{b'_x, b'_y\}$. Then, the total values of the velocity and the magnetic induction

$$\mathbf{U}=\mathbf{U}^c(y)+\mathbf{u}'(x, y, t), \quad \mathbf{B}=\mathbf{B}^c(y)+\mathbf{b}'(x, y, t) \tag{2}$$

must satisfy a system of the equations of magnetic hydrodynamics [7]

$$\frac{\partial \mathbf{U}}{\partial t}+(\mathbf{U}\cdot\nabla)\cdot\mathbf{U}-\mathrm{Al}^2(\mathbf{B}\cdot\nabla)\cdot\mathbf{B}=-\nabla P_m+\frac{\triangle \mathbf{U}}{\mathrm{Re}},$$

$$\frac{\partial \mathbf{B}}{\partial t}+(\mathbf{U}\cdot\nabla)\cdot\mathbf{B}-(\mathbf{B}\cdot\nabla)\cdot\mathbf{U}=\frac{\triangle \mathbf{B}}{\mathrm{Re}_m}, \quad \nabla\mathbf{U}=\nabla\mathbf{B}=0, \tag{3}$$

where Re is the ordinary Reynolds number; Al is the Alfvén number [8]. Here, the generalized pressure P_m is equal to the sum of the ordinary and magnetic pressures; $P_m = P + \frac{1}{2}\mathrm{Al}^2B^2$ may be represented in the form

$$P_m=P^c{}_m+p'_m, \tag{4}$$

where $P^c{}_m$ relates to the mean flow, and p'_m to unsteady-state perturbations.

Following [3-5], we set

$$u'_x;\ u'_y;\ b'_x;\ b'_y;\ p'_m=[u_x;\ u_y;\ b_x;\ b_y;\ p_m e^{i\alpha(x-ct)}+u^*_x;$$
$$u^*_y;\ b^*_x;\ b^*_y;\ p^*_m e^{-i\alpha(x-ct)}], \tag{5}$$

where the sign * denotes the operation of complex conjugation; α is the projection of the wave vector on the x axis; αc is the frequency of the perturbations; all the quantities u_x, u_y, ..., p^*_m are functions only of y. Substituting expressions (2), (4) into system (3), and taking account of (5), after averaging with respect to $x=2\pi/\alpha$ we can obtain the following system of equations for the amplitudes of the perturbations:

$$\frac{1}{2}\{[u^*_yDu_x+u_yDu^*_x]-\mathrm{Al}^2[b^*_yDb_x+b_yDb^*_x]\}=-\frac{\partial P^c{}_m}{\partial x}+\frac{D^2U^c}{\mathrm{Re}}+\mathrm{Al}^2DB^c, \tag{6}$$

$$Lu_x=i\alpha p_m+u_yDU^c-\mathrm{Al}^2(Nb_x+b_yDB^c), \tag{7}$$

$$\frac{1}{2}\{D|u_y|^2+i\alpha(u^*_xu_y-u^*_yu_x)-\mathrm{Al}^2[D|b_y|^2+i\alpha(b^*_xb_y-b^*_yb_x)]\}=-\frac{\partial P^c{}_m}{\partial y}, \tag{8}$$

$$Lu_y=Dp_m-\mathrm{Al}^2Nb_y, \tag{9}$$

$$\frac{1}{2}D(u_yb^*_x+u^*_yb_x-u_xb^*_y-u^*_xb_y)=DU^c+\frac{D^2B^c}{\mathrm{Re}_m}, \tag{10}$$

$$L_mb_x=u_yDB^c-b_yDU^c-Nu_x, \tag{11}$$

$$L_mb_y=-Nu_y, \quad i\alpha u_x+Du_y=0, \quad i\alpha b_x+Db_y=0; \tag{12), (13), (14}$$

here $L\equiv[(D^2-\alpha^2)/\mathrm{Re}]-i\alpha(U^c-c)$; $L_m\equiv[(D^2-\alpha^2)/\mathrm{Re}_m]-i\alpha(U^c-c)$; $D\equiv d/dy$; $N\equiv D+i\alpha B^c$. Eliminating p_m from Eqs. (7) and (9), and taking account of (13) and (14), we can obtain

$$L(Du_x-i\alpha u_y)=u_yD^2U^c-\mathrm{Al}^2[N(Db_x-i\alpha b_y)+b_yD^2B^c]. \tag{15}$$

Combining Eqs. (9), (13)-(15), we have

$$L_1u_y\equiv[(U^c-c)(D^2-\alpha^2)-D^2U^c+\frac{i}{\alpha\,\mathrm{Re}}(D^2-\alpha^2)^2]u_y=\frac{\mathrm{Al}^2}{i\alpha}[N(D^2-\alpha^2)-i\alpha D^2B^c]b_y. \tag{16}$$

In what follows, we shall assume that $\mathrm{Re}_m \ll 1$; under these circumstances, from Eqs. (13) and (16), there follows an equation containing only u_y:

$$L_1u_y=\frac{i\,\mathrm{Ha}^2}{\alpha\,\mathrm{Re}}D^2u_y. \tag{17}$$

This equation, which is the Orr-Sommerfeld equation for plane, two-dimensional perturbations [1, 2, 7], must be solved under the usual conditions of the reversion to zero of the velocity perturbations at the solid surfaces of the channel.

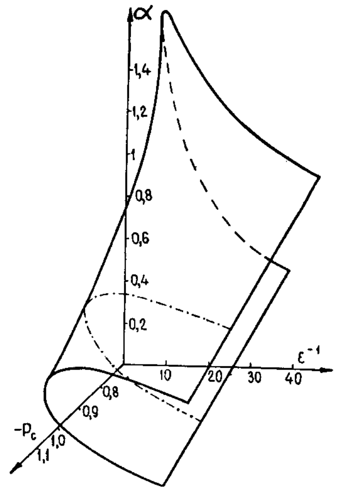

Fig. 1

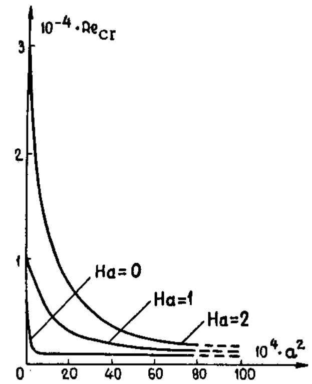

Fig. 2

Fig. 1. Surface of the eigenvalues of α, $-P_c$, ε^{-1} of the solution of Eq. (39); the dot-dash line corresponds to the curve of neutral stability for the value Ha = 1, obtained in [7].

Fig. 2. Dependence of the critical values of the Reynolds number on the amplitude of the perturbation.

The distribution of the velocity of the mean flow $U^c(y)$ and of the induced magnetic field $B^c(y)$ may be found from Eqs. (6) and (10) which, after transformations taking account of (13) and (14), assume the form

$$\frac{D^2U^c}{\text{Re}}+\text{Al}^2DB^c-\frac{\partial P^c{}_m}{\partial x}=\frac{D}{2}\left[u^*{}_xu_y+u^*{}_yu_x-\text{Al}^2\left(b^*{}_xb_y+b^*{}_yb_x\right)\right], \tag{18}$$

$$\frac{D^2B^c}{\text{Re}_m}+DU^c=\frac{D}{2}\left[b^*{}_xu_y+u^*{}_yb_x-b^*{}_yu_x-u^*{}_xb_y\right]. \tag{19}$$

Setting

$$\begin{gathered}u^*{}_xu_y+u^*{}_yu_x=2a^2\tau_1;\quad b^*{}_xb_y+b^*{}_yb_x=2a_m{}^2\tau_2;\\ b^*{}_xu_y+u^*{}_yb_x-b^*{}_yu_x-u^*{}_xb_y=2aa_m\tau_3\end{gathered} \tag{20}$$

(α is a constant factor in the expression for the amplitude of the velocity perturbation; α_m is a constant factor in the expression for the amplitude of the perturbation of the induced field), after integration of Eqs. (18), (19), we can find

$$\begin{gathered}U^c(y)=1+T(y)-\frac{\text{ch Ha}\,y-1}{\text{ch Ha}-1}\left[1+T(1)\right],\\ T(y)=\int_0^y\left[\text{Re}\left(a^2\tau_1-\text{Al}^2a_m{}^2\tau_2\right)\text{ch Ha}\,(y-\xi)\;-\text{Ha}\,aa_m\tau_3\,\text{sh Ha}\,(y-\xi)\right]d\xi,\\ B^c(y)=\frac{\text{Re}_m}{\text{Ha}}\left[\frac{\text{sh Ha}\,y-y\,\text{sh Ha}}{\text{ch Ha}-1}+T_m(y)-\frac{\text{sh Ha}\,y}{\text{sh Ha}}\,\text{sign}\,y\cdot T_m(1)\right],\end{gathered} \tag{21}$$

$$T_m(y)=-\int_0^y\left[\text{Re}\left(a^2\tau_1-\text{Al}^2a_m{}^2\tau_2\right)\text{sh Ha}\,(y-\xi)\;-\text{Ha}\,aa_m\tau_3\,\text{ch Ha}\,(y-\xi)\right]d\xi. \tag{22}$$

It is not difficult to verify that expressions (20)-(22) satisfy the conditions

$$\begin{gathered}U^c(\pm1)=B^c(\pm1)=B^c(0)=0;\quad U^c(0)=1;\\ \tau_1(0)=\tau_2(0)=\tau_3(0)=0.\end{gathered} \tag{23}$$

With $Re_m \ll 1$, the perturbations of the magnetic field can be neglected in comparison with the perturbations of the velocity and, consequently, in the expressions for T and T_m, we can discard terms containing $a_m^2\tau_2$ and $aa_m\tau_3$.

We now proceed to the solution of Eq. (17).

For even perturbations of u_y, the most dangerous from the point of view of loss of stability, Eq. (17) must be solved with the conditions [1]

$$u_y(-1) = Du_y(-1) = Du_y(0) = D^3u_y(0) = 0. \tag{24}$$

In accordance with the usual procedure, the two first independent parts of the solution of Eq. (17), $u_y^{(1)}$, $u_y^{(2)}$, must be obtained from solution of the nonviscous equation

$$[(U^c - c)(D^2 - \alpha^2) - D^2U^c]\, u_y = 0. \tag{25}$$

Representing the solutions of Eqs. (25) in the form of power series with respect to $y - y_c$, where y_c is the point at which $U^c(y_c) = c$, we have

$$u_y^{(1)} = (y - y_c)\sum_{k=0}^{\infty} a_k (y - y_c)^k; \quad u_y^{(2)} = P_c \ln (y - y_c)\, u_y^{(1)} + \sum_{k=0}^{\infty} b_k (y - y_c)^k, \tag{26}$$

where

$$\begin{gathered} a_0 = 0; \quad a_1 = b_0 = 1; \quad a_2 = b_1 = A_2; \\ a_{k+1} = A_{k+1} + \frac{1}{k(k+1)}\left\{\alpha^2 A_{k-1} + \sum_{j=2}^{k}[\alpha^2 A_{k-j} + (k+1)(k+2-2j)A_{k-j+2}]\, a_j\right\} \\ (k \geqslant 2); \\ b_k = a_{k+1} - \sum_{j=1}^{k} \frac{\gamma_{j+1}}{j} a_{k-j}, \\ \gamma_l = -\sum_{j=1}^{l}\left(\sum_{k=0}^{j+2} a_k a_{j-k+2}\right)\gamma_{l-j}\,(l \geqslant 1), \quad A_j \equiv \frac{D^j U_{0x}(y_c)}{DU_{0x}(y_c)}. \end{gathered} \tag{27}$$

In the solutions (26) and (27), the coefficients α_k, b_k are determined under the assumption that, for $U^c(y)$, it is sufficient to take $U_{0x}(y)$, determined by the distribution (1) [3, 4]; the value of $P_c \equiv D^2U^c(yc)/DU^c(yc)$ in the expression for the solution $u_y^{(2)}$ (26) will be determined as an eigenvalue in the solution of the corresponding secular equation.

The general even solution of the nonviscous Eq. (25) must be written in the form

$$u_y^{(n)} = A u_y^{(1)} + u_y^{(2)}, \tag{28}$$

where the arbitrary constant A must be determined from the condition

$$Du_y^{(n)}(0) = 0. \tag{29}$$

The second pair of independent partial solutions of Eq. (17), $u_y^{(3)}$, $u_y^{(4)}$, the so-called viscous solutions, can be found after the introduction of a new independent variable η

$$\eta \equiv (y - y_c)/\varepsilon, \quad \varepsilon = [\alpha \,\mathrm{Re}\, DU^c(y_c)]^{-1/3}. \tag{30}$$

Then, representing the solution of Eq. (17) in the form of a series in powers of ε, and limiting ourselves to the zero approximation, for determining $u_y^{(3)}$ and $u_y^{(4)}$, we have the equation

$$\frac{d^4u_y}{d\eta^4} - i\eta\frac{d^2u_y}{d\eta^2} = 0, \tag{31}$$

two solutions of which can be represented in the form [1]

$$u_y^{(3)} = \int_{\infty}^{\eta} d\eta \int_{\infty}^{\eta} \sqrt{\eta} H^{(1)}_{1/3}\left[\frac{2}{3}(i\eta)^{3/2}\right] d\eta; \tag{32}$$

$$u_y^{(4)} = \int_{-\infty}^{\eta} d\eta \int_{-\infty}^{\eta} \sqrt{\eta} H^{(2)}_{1/3}\left[\frac{2}{3}(i\eta)^{3/2}\right] d\eta, \tag{33}$$

where $H^{(1)}_{1/3}$ and $H^{(2)}_{1/3}$ are, respectively, Hankel functions of the 1st and 2nd kind, of the $1/3$ order. Two other solutions of Eq. (31), refined by taking account of the first approximation, may be compared with the solutions $u_y^{(1)}$ and $u_y^{(2)}$ (26) of the nonviscous equation (25); this permits isolating the appropriate branch in the solution $u_y^{(2)}$ (26), bypassing the point $y = y_c$.

$$-\frac{7\pi}{6} < \arg\eta < \frac{\pi}{6}.$$

It must also be noted that $Du_y^{(2)} \to \infty$ at $y \to y_c$; therefore, the solution $u_y^{(2)}$ must be written taking account of the so-called viscous correction to the solution of the nonviscous equation, as a result of which the solution $u_y^{(2)}$, instead of expression (26), is written in the form

$$u_y^{(2)} = \frac{P_c}{\eta}[S(\eta) + \eta \ln|\varepsilon|]\,u_y^{(1)} + \sum_{k=0}^{\infty} b_x (y - y_c)^k \tag{34}$$

(the value of the function $S(\eta)$ is tabulated in [9]). If we discard the rapidly rising partial solution $u_y^{(4)}$, the expression for the general solution of Eq. (17) can be written in the form

$$u_y = u_y^{(n)} + B u_y^{(3)}, \tag{35}$$

where the arbitrary constant B is determined from the boundary conditions (24).

After the expression for the general solution u_y (35) has been obtained, we can determine $a^2\tau_1$. Actually, from the definition of $\alpha^2\tau_1$ (20) and Eq. (13), we have

$$a^2\tau_1 = \frac{i}{\alpha}(u^*_y D u_y - u_y D u^*_y), \tag{36}$$

after which, knowing $a^2\tau_1$, we can proceed to plotting of the profile of the mean velocity (21), and to calculation of its derivatives at the critical point $y = y_c$:

$$DU^c(y_c) = Q_1 + Q_2 a^2 \alpha \,\mathrm{Re}, \qquad D^2U^c(y_c) = P_1 + P_2 a^2 \alpha\,\mathrm{Re}, \tag{37), (38}$$

where Q_1, Q_2, P_1, P_2 are constants, calculated taking account of (36).

From the expression for the general solution u_y (35) and conditions (24), we can obtain the secular equation

$$-\frac{1}{1+y_c}\frac{u_y^{(H)}}{Du_y^{(H)}} - \frac{1}{\eta}\frac{u_y^{(3)}}{Du_y^{(3)}} \equiv \Psi(\zeta), \tag{39}$$

where $\Psi(\zeta)$ is a tabulated Tietjens function [1], and $\zeta = (1 + y_c) \times [\alpha \mathrm{Re} DU^c(y_c)]^{1/3}$. Solving Eq. (39) using the Tollmin graphical method [1], with given values of the parameter of the problem i.e., the Ha number, we can find the eigenvalues of y_c, P_c, α, and ε. We note that the above determination of P_c as an eigenvalue of the secular equation (39), in the general case does not coincide with $D^2U_{0x}(y)/DU_{0x}(y)$, calculated using the expression for the velocity distribution of Hartmann flow (1). In three dimensions (α, $-P_c$, ε^{-1}) the set of eigenvalues of the secular equation forms a hypersurface, on which, in particular, there is a neutral curve, corresponding to infinitely small perturbations. Figure 1 shows the hypersurface, constructed for the case Ha = 1.

From expressions (37), (38), and the parameters of any point of the hypersurface (α, $-P_c$, ε^{-1}), we can obtain an expression for the corresponding Reynolds number

$$\mathrm{Re} = \frac{P_c Q_2 - P_2}{\alpha (Q_1 P_2 - P_1 Q_2)\varepsilon^3} \tag{40}$$

and α is a constant factor in the expression for the amplitude of the velocity perturbation

$$a^2 = \frac{\varepsilon^{-3} - \alpha\,\mathrm{Re}\,Q_1}{\alpha^2\,\mathrm{Re}^2\,Q_2}. \tag{41}$$

Figure 2, on the basis of the results of calculations carried out in accordance with the above-described scheme and performed in a BÉSM-6 digital computer, gives plots of the dependence of Re_{cr} on a^2. As was to be expected, with an increase in a, the values of Re_{cr} decrease substantially. In addition, with a rise in the intensity of the magnetic field, there is a rapid rise in the values of Re_{cr}, corresponding to infinitely small perturbations, and a slowing down of the rise in the values of Re_{cr}, corresponding to perturbations of finite amplitude. This fact must obviously be compared with the results of [10, 11], in which note is taken of the ability of a magnetic field to suppress the turbulent transfer of energies between perturbations of different scales: with a rise in the intensity of the field, ever smaller scales become forbidden for the transfer of energy upward over the spectrum.

We note that the method discussed in the present article can be extended to the case of three-dimensional perturbations of finite amplitudes, when the wave vector of the perturbations has projections on the x and z axes. A corresponding study of the hydrodynamic stability of Poiseuille flow was made in [12]. As preliminary calculations have shown, for small Hartmann numbers (Ha = 0-5), in the case of perturbations of arbitrary amplitude, the Squire theorem does not hold.

A study of the magnetohydrodynamic stability of gradient Hartmann flow and of shear Couette Flow, within the framework of the nonlinear theory, has been made using the energy method [13]. However, in the investigation of hydrodynamic stability, the energy method gives a too low value of the critical Reynolds number; obviously, this fact explains the divergence between the results of [13] and the present article.

LITERATURE CITED

1. Lin Chia-ch'iao, Hydrodynamic Stability, Cambridge Univ. Press.
2. H. Schlichting, The Development of Turbulence [Russian translation], Izd. Inostr. Lit., Moscow (1962).
3. D. Meksyn and J. T. Stuart, Proc. Roy. Soc., A208, 517 (1951).
4. J. T. Stuart, ZAMM, Sonderheft "Tagung des Fachausschesses der GAMM für Stromungsforschung in Göttingen, 1955," (1956), p. 32.
5. J. T. Stuart, J. Fluid Mech., 4, No. 1, 1 (1958).
6. W. C. Reynolds and M. C. Potter, J. Fluid. Mech., 27, No. 3, 465 (1967).
7. R. C. Lock, Proc. Roy. Soc., A233, 105 (1955).
8. Pai Shih-i, Magnetogasdynamics and Plasma Dynamics, Springer-Verlag (1962).
9. H. Holstein, ZAMM, 30, No. 1, 25 (1950).
10. L. G. Kit, "Study of the effect of a magnetic field on the velocity- and unsteady-state velocity structures of the flows of an electrically conducting liquid," Candidate's Dissertation, Riga (1971).
11. G. G. Branover, Yu. M. Gel'fgat, L. G. Kit, and A. B. Tsinober, Izv. Akad. Nauk SSSR, Mekhan. Zhidk. i Gaza, No. 2, 35 (1970).
12. D. Meksyn, Ztschr. Phys., 2, 178, 159 (1964).
13. A. M. Sagalakov and V. N. Shtern, Izv. Akad. Nauk SSSR, Mekhan. Zhidk. i Gaza, 4, No. 3 (1971).