HYDRODYNAMIC STABILITY OF A GRADIENT FLOW OF A CONDUCTING FLUID WITH A RHEOLOGICAL POWER LAW IN A TRANSVERSE MAGNETIC FIELD

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We consider the problem of the hydrodynamic stability of a steady Hartmann flow of a conducting fluid with a rheological power law with respect to plane infinitesimal perturbations. Neutral-stability curves are obtained for certain values of the defining parameters.

There have recently been published a number of works [1-5], which investigate MHD flows of conducting non-Newtonian fluids, which have more complicated mechanical properties than an incompressible Newtonian fluid. These investigations enabled us to detect in such flows a number of effects related to the presence of non-Newtonian properties in the fluid. In particular, interesting nonlinear effects related to the formation of bands localized in space having nonzero shear stresses were detected in an investigation of MHD flows of conducting fluids with a rheological power law in a transverse magnetic field [3-5]. For such media, the rheological equation relating the stress-tensor deviator s_{ij} and the deformation rate tensor f_{ij} has the form

$$s_{ij}=2k_n\omega^{n-1}f_{ij} \qquad (i,\ j=1,\ 2,\ 3), \tag{1}$$

where k_n, $n>0$, is the rheological constant of the medium, and $\omega=\sqrt{2f_{ij}f_{ij}}$ is the second invariant of the deformation velocity tensor. Media having $n>1$ are called dilatant fluids, and media with $n<1$ are called pseudoplastic fluids; the case $n=1$ corresponds to a viscous Newtonian fluid. We should note that the quantity $\eta_n=k_n\omega^{n-1}$ in [1] is effectively a viscous fluid, which for $n\neq 1$ is not a constant for various flow regions, and for dilatant fluids, when $n>1$, vanishes at points at which $\omega=0$.

There is interest in investigating the stability of MHD flows of conducting non-Newtonian fluids with a rheological power law. We note that investigations on the hydrodynamic stability of a number of plane flows were carried out in [6,7] for nonconducting non-Newtonian fluids; furthermore, the stability of a shear MHD flow of a conducting dilatant fluid was investigated in [8].

In the present study we investigate the hydrodynamic stability relative to infinitesimal two-dimensional perturbations of plane, steady, gradient Hartmann flow of a conducting fluid with a rheological power law in a channel in a transverse homogeneous magnetic field.

Let an isotropic conducting incompressible fluid with rheological law (1) undergo steady motion in a plane channel under the action of a constant pressure gradient $\partial p/\partial x=\mathrm{const}<0$. The external magnetic field with induction $B_y=B_0=\mathrm{const}$ is perpendicular to the nonconducting walls of the channel $y=\pm L$, and the constant electric field $\mathbf{E}_0$ is directed along the z axis.

The system of equations of magnetohydrodynamics then will have the form

$$\varrho\left(\frac{\partial}{\partial t}+\mathbf{u}\nabla\right)\mathbf{u}=-\nabla p+\nabla s+\frac{1}{\mu_0}[[\nabla\times\mathbf{B}]\times\mathbf{B}];$$

$$\frac{\partial\mathbf{B}}{\partial t}=[\nabla\times[\mathbf{u}\times\mathbf{B}]]+\frac{1}{\mu_0\sigma}\Delta\mathbf{B}, \qquad \nabla\mathbf{u}=\nabla\mathbf{B}=0; \tag{2}$$

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here s is determined from Eq. (1); the remaining notation is obvious.

As was shown in [3], solution of system (2) for such a formulation of the problem allows us to find the steady-state velocity profile in the channel $\mathbf{U}[U_{0x}(y), 0, 0]$ in the form of an analytical implicit dependence $y = f(U_{0x})$, which in dimensionless form for the lower half of the channel $-1 \leq y \leq 0$ can be written in the form

$$yM = -\frac{n+1}{2\gamma n}(v^2-\gamma^2)^{\frac{n}{n+1}} F\left[\frac{1}{2}, \frac{n}{n+1}; \frac{2n+1}{n+1}; 1-\frac{v^2}{\gamma^2}\right], \tag{3}$$

where

$$v \equiv 1-U_{0x}, \quad \gamma = 1-U_{\max}, \quad U_{0x} \leqslant U_{\max} \leqslant 1, \quad M = \left(\frac{\mathrm{Ha}_n{}^2}{2}\frac{n+1}{n}\right)^{\frac{n}{n+1}},$$

$\mathrm{Ha}_n{}^2 = (\sigma B_0{}^2 L^{n+1} V^{1-n})/k_n$ is the square of the generalized Hartmann number, and $F[k_1, k_2; m; \theta]$ is the Gauss hypergeometric function. Writing Eq. (3) for the characteristic values of length and velocity, respectively, we take the channel half-width L and the quantity $V = P_E/\sigma B_0{}^2$, where $P_E = -\partial p/\partial x + \sigma E_0 B_0$ is the effective pressure gradient, taking into account the effect of the Ampere force. The quantity γ in (3) characterizes the maximum velocity in the channel and is determined from the transcendental equation

$$M = \frac{n+1}{2\gamma n}(1-\gamma^2)^{\frac{n}{n+1}} F\left[\frac{1}{2}, \frac{n}{n+1}; \frac{2n+1}{n+1}; 1-\frac{1}{\gamma^2}\right]. \tag{4}$$

Equation (3) with account of (4) determines the velocity profile in the channel for each flow regime of a pseudoplastic fluid ($n < 1$), and for a dilatant fluid ($n > 1$) only for the case in which

$$\mathrm{Ha}_n{}^2 < \mathrm{Ha}_*{}^2 = \frac{2n}{n-1}\left(\frac{n+1}{n-1}\right)^n. \tag{5}$$

If $\mathrm{Ha}_n{}^2 \geq \mathrm{Ha}_*{}^2$, then at the center of the channel there appears a quasisolid zone, and the velocity distribution in the channel takes the form [3]

$$U_{0x}(y) = \begin{cases} 1-\left[1-M\dfrac{n-1}{n+1}(1+y)\right]^{\frac{n+1}{n-1}} & -1 \leqslant y \leqslant y_0, \\ 0 & y_0 \leqslant y \leqslant 0. \end{cases} \tag{6}$$

The size of the quasisolid zone $|y_0|$ is determined from the equation

$$|y_0| = 1 - \frac{n+1}{M(n-1)}. \tag{7}$$

We consider the question of the hydrodynamic stability of flows (3) and (6) with respect to infinitesimal plane perturbations propagating in the x, y plane. Linearizing (2) in the usual manner and representing the y projection of the dimensionless perturbations of velocity $u'_y(x, y, t)$ and of magnetic field $b'_y(x, y, t)$ in the form

$$u'_y = \psi(y)\exp[i\alpha(x-ct)], \quad b'_y = \varphi(y)\exp[i\alpha(x-ct)], \tag{8}$$

where α is the wave number, αc is the complex perturbation frequency, we can obtain the following system of two differential equations for determining the functions $\psi(y)$ and $\varphi(y)$:

$$\begin{cases} L\psi = \dfrac{\mathrm{Ha}_n{}^2}{\mathrm{Re}_n \mathrm{Re}_m}\left[(D^2-\alpha^2)\left(B_{0x}-\dfrac{iD}{\alpha}\right)-(D^2 B_{0x})\right]\varphi; \\ \left(B_{0x}-\dfrac{iD}{\alpha}\right)\psi = \left[(U_{0x}-c)+\dfrac{i}{\alpha \mathrm{Re}_m}(D^2-\alpha^2)\right]\varphi. \end{cases} \tag{9}$$

Here

$$L \equiv (U_{0x}-c)(D^2-\alpha^2) - D^2 U_{0x} + \frac{i[(DU_{0x})^2]^{\frac{n-3}{2}}}{\alpha \mathrm{Re}_n}\langle (DU_{0x})^2 n (D^2-\alpha^2)^2 +$$
$$+ (n-1)\{2n(DU_{0x})(D^2U_{0x})D^3 + [4\alpha^2(DU_{0x})^2 + n(DU_{0x})(D^3U_{0x}) +$$

$$+n(n+2)(D^2U_{0x})^2]D^2+2(n-2)\alpha^2(DU_{0x})(D^2U_{0x})D+\alpha^2 n[(DU_{0x})\times$$
$$\times(D^3U_{0x})+(n-2)(D^3U_{0x})^2]\}\rangle; \qquad D\equiv\frac{d}{dy}; \qquad \mathrm{Re}_n=\frac{\varrho V^{2-n}L^n}{k_n}$$

is the generalized Reynolds number for a fluid with rheological power law (1); $\mathrm{Re_m}=\mu_0\sigma VL$ is the magnetic Reynolds number; B_{0x} is the field induced by the flow of the medium.

If $\mathrm{Re_m}\ll 1$, then system (9), after we neglect small terms, can be reduced to a single differential equation in $\psi(y)$:

$$L\psi=\frac{i\,\mathrm{Ha}_n}{\alpha\,\mathrm{Re}_n}D^2\psi. \tag{10}$$

For n=1, Eq. (10) becomes the Lock equation, which is used for investigating the flow stability of a conducting viscous Newtonian fluid in a transverse magnetic field, when $\mathrm{Re_m}\ll 1$ or $\mathrm{Re_m}\sim 1$ [9-12].

If $\alpha\ \mathrm{Re_n}$ is large, then the function $\psi(y)$ in (10) can be represented by the asymptotic expansion

$$\psi(y)=\sum_{k=0}^{\infty}\frac{\psi^{(k)}(y)}{(\alpha\,\mathrm{Re}_n)^k}, \tag{11}$$

and ψ_1 and ψ_2, the first pair of independent solutions of Eq. (10), must be found from the nonviscous equation

$$[(U_{0x}-c)(D^2-\alpha^2)-(D^2U_{0x})]\psi=0, \tag{12}$$

which is an equation for determining the zero approximation $\psi^{(0)}(y)$ in expansion (11). The solution of Eq. (12) can be represented in the form of power series in $y-y_c$, where y_c is the point at which $U_{0x}(y_c)=c$;

$$\psi_1=\sum_{j=0}^{\infty}a_j(y-y_c)^j; \qquad \psi_2=\psi_1 b_1\ln(y-y_c)+b_0+$$
$$+\sum_{j=1}^{\infty}\left[a_{j+1}-\sum_{k=1}^{j}\frac{b_{k+1}}{k}a_{j-k}\right](y-y_c)^j; \qquad a_0=0; \qquad a_1=b_0=1;$$
$$a_2=A_2; \qquad a_{m+1}=A_{m+1}+\frac{1}{m(m+1)}\left\{\alpha^2A_{m-1}+\sum_{j=2}^{m}[\alpha^2A_{m-j}+\right. \tag{13}$$
$$\left.+(m^2-2mj+3m-2j+2)A_{m-j+2}]a_j\right\} \quad (m\geqslant 2);$$
$$A_j=\frac{D^jU_{0x}(y_c)}{j!DU_{0x}(y_c)}; \qquad b_l=-\sum_{j=1}^{l}\left(\sum_{k=0}^{j+2}a_k a_{j-k+2}\right)b_{l-j} \quad (l\geqslant 1).$$

For calculating $D^jU_{0x}(y)$ in (13) for flow regimes of a conducting dilatant fluid with a quasisolid zone (6) we must use directly the analytical expression (6), whereas for flow regimes described by Eq. (3), we can use the equation [3]

$$DU_{0x}(y)=\begin{cases} M[(1-U_{0x})^2-\gamma^2]^{\frac{1}{n+1}} & -1\leqslant y<y_0, \\ 0 & y_0\leqslant y\leqslant 0. \end{cases} \tag{14}$$

Another pair of independent particular solutions of Eq. (10) ψ_3 and ψ_4 are in the form

$$\psi(y)=\exp\left(\int g\,dy\right), \qquad g=\sum_{m=0}^{\infty}(\alpha\,\mathrm{Re}_n)^{\frac{1-m}{2}}g_m. \tag{15}$$

By substituting (15) into Eq. (10), we can determine

$$g_0=\pm\sqrt{\frac{i(U_{0x}-c)}{n(DU_{0x})^{n-1}}}, \qquad g_1=-\frac{5DU_{0x}}{4(U_{0x}-c)}+\frac{(n-1)D^2U_{0x}}{4DU_{0x}}, \tag{16}$$

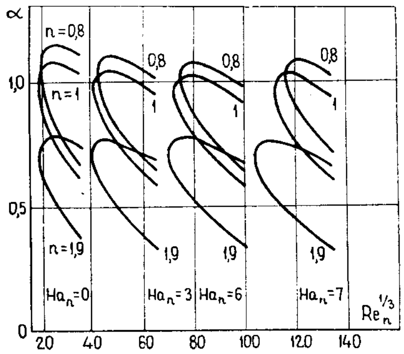

Fig. 1

as a result of which we can obtain

$$\psi_{3,4}=(U_{0x}-c)^{-\frac{5}{4}}(DU_{0x})^{\frac{n-1}{4}}\exp\left[\mp\int_{y_c}^{y}\sqrt{\frac{i\alpha\,\mathrm{Re}_n\,(U_{0x}-c)}{n(DU_{0x})^{n-1}}}\,dy\right]. \quad (17)$$

Solutions ψ_3 and ψ_4 near $y=y_c$ are found directly from Eq. (10) with introduction of the new variable

$$\eta=\frac{y-y_c}{\varepsilon}, \quad \varepsilon=(\alpha\,\mathrm{Re}_n)^{-1/3}. \quad (18)$$

If we seek solutions $\psi(y)\equiv\chi(\eta)$ in the form of a series in powers of ε: $\chi(\eta)=\sum_{k=0}^{\infty}\varepsilon^k\chi^{(k)}$, then after equating coefficients of the same powers of ε we can obtain

$$\begin{aligned}\chi_1^{(0)}&=\eta; \quad \chi_3^{(0)}=\int_{\infty}^{\eta}d\eta\int_{\infty}^{\eta}\sqrt{\eta}H_{1/3}^{(1)}\left[\frac{2}{3}(ia\eta)^{3/2}\right]d\eta;\\ \chi_2^{(0)}&=1; \quad \chi_4^{(0)}=\int_{-\infty}^{\eta}d\eta\int_{-\infty}^{\eta}\sqrt{\eta}H_{1/3}^{(2)}\left[\frac{2}{3}(ia\eta)^{3/2}\right]d\eta;\end{aligned} \quad (19)$$

here $H_{1/3}^{(1)}$ and $H_{1/3}^{(2)}$ are Hankel functions, and $a=\sqrt[3]{\frac{[DU_{0x}(y_c)]^{2-n}}{n}}$. The asymptotic behavior of the Hankel functions enables us to identify the solutions $\psi_{1,2,3,4}$ in correspondence with the solutions $\chi_{1,2,3,4}$, and also to determine the necessary branch for bypassing the point y_c:

$$-\frac{7\pi}{6}<\arg(y-y_c)<\frac{\pi}{6}.$$

For an investigation of the flow stability of a dilatant fluid (6) we should investigate the behavior of perturbations in the quasisolid zone $y_0\leq y\leq 0$, in which shearing stresses are everywhere absent and the medium moves as an ideal fluid ($U_{0x}=\mathrm{const}$). The equation for the perturbations $\psi(y)$ in an ideal fluid in the zone $y_0\leq y\leq 0$ is written in the form

$$(D^2-\alpha^2)\psi=0. \quad (20)$$

The solution of Eq. (20), which is even with respect to $y=0$, is the function

$$\psi(y)=A\,\mathrm{ch}\,\alpha y, \quad A=\mathrm{const} \quad (21)$$

(investigation of the even solutions of Eqs. (10) and (21) with respect to $y=0$ leads eventually to smaller critical values Re_n^{cr} than for the investigation of the odd solutions).

For flow regimes of dilatant fluids ($n>1$) described by Eq. (3), as a result of which $DU_{0x}(0)=0$, it is evident that $\psi_{3,4}(0)=0$, and the derivatives $D\psi_{3,4}(0)$ are singular. The singularity in the value of the derivative of the general solution of Eq. (10)

$$\psi(y)=C_1\psi_1+C_2\psi_2+C_3\psi_3+C_4\psi_4, \quad (22)$$

calculated at the point $y=0$, can be removed if the following condition is satisfied [8]:

$$C_3D\psi_3(0)+C_4D\psi_4(0)=0. \quad (23)$$

The conditions for the general solution (22) being nontrivial, written using the conditions of adhesion at the point $y=-1$,

$$\begin{aligned}C_1\psi_1(-1)+C_2\psi_2(-1)+C_3\psi_3(-1)+C_4\psi_4(-1)&=0,\\ C_1D\psi_1(-1)+C_2D\psi_2(-1)+C_3D\psi_3(-1)+C_4D\psi_4(-1)&=0;\end{aligned} \quad (24)$$

the conditions for the evenness of the perturbations at the point y = 0

$$C_1D\psi_1(0)+C_2D\psi_2(0)+C_3D\psi_3(0)+C_4D\psi_4(0)=0\ , \tag{25}$$

and conditions (23) lead to a secular equation, which after its terms have been estimated in order of magnitude, can be represented in the form

$$\frac{D\psi_3(-1)}{\psi_3(-1)}=\frac{\begin{vmatrix} D\psi_1(-1) & D\psi_2(-1) \\ D\psi_1(0) & D\psi_2(0) \end{vmatrix}}{\begin{vmatrix} \psi_1(-1) & \psi_2(-1) \\ D\psi_1(0) & D\psi_2(0) \end{vmatrix}}\ . \tag{26}$$

The left side of Eq. (26) is expressed in terms of the tabulated function of Tietjens [13]; the right side must be calculated using the solutions of ψ_1 and ψ_2 constructed above (13).

Investigation of the stability of a Hartmann flow of a pseudoplastic fluid eventually also reduces to finding the eigenvalues of the secular equation (26), which in this case follows from the requirement that the singularity in the general solution (22) and its first derivative be removed at the point $y=0$ for $n<1$.

For the case of flow regimes of dilatant fluids ($n>1$) described by the distribution (6), the viscous solution $\psi_3(y_0)=0$, and the derivative $D\psi_4(y_0)$ is singular. The singularity in the value of the derivative of the general solution $\psi(y)$ $(-1\le y\le y_0)$ (22) is removed at the point $y=y_0$ when the following condition is satisfied [8]:

$$C_3D\psi_3(y_0)+C_4D\psi_4(y_0)=0. \tag{27}$$

From the requirement of continuity of the solutions (22) and (8), and also of their first derivatives, at the point $y=y_0$, taking account of condition (27) and the fact that $\psi_3(y_0)=0$, we have

$$C_1[D\psi_1(y_0)-\alpha\,\mathrm{th}\,(\alpha y_0)\psi_1(y_0)]+C_2[D\psi_2(y_0)-\alpha\,\mathrm{th}\,(\alpha y_0)\psi_2(y_0)]=0. \tag{28}$$

Conditions (27) and (28), together with the adhesion conditions (24), allow us to write the condition for nontriviality of the general solution (22), which after carrying out estimates of its terms to order of magnitude, and neglecting small quantities, reduces to the following secular equation:

$$\frac{D\psi_3(-1)}{\psi_3(-1)}=\frac{\begin{vmatrix} D\psi_1(-1) & D\psi_2(-1) \\ D\psi_1(y_0)-\alpha\,\mathrm{th}\,(\alpha y_0)\psi_1(y_0) & D\psi_2(y_0)-\alpha\,\mathrm{th}\,(\alpha y_0)\psi_2(y_0) \end{vmatrix}}{\begin{vmatrix} \psi_1(-1) & \psi_2(-1) \\ D\psi_1(y_0)-\alpha\,\mathrm{th}\,(\alpha y_0)\psi_1(y_0) & D\psi_2(y_0)-\alpha\,\mathrm{th}\,(\alpha y_0)\psi_2(y_0) \end{vmatrix}}\ . \tag{29}$$

Figure 1 shows curves of the neutral stability on the α, Re_n plane, constructed from the numerical solution of Eqs. (26) and (29) for several values of n and Ha_n.

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