ELECTRODYNAMIC FLOW OF A VISCOPLASTIC FLUID

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Steady-state electrodynamic flow is considered for Shvedov–Bingham plastic in a plane channel having dielectric walls, in a longitudinal, homogeneous electric field, and acted upon by a pressure gradient which is constant in time. Various modes of medium flow are investigated. The distribution of shear stress and the medium velocity in the channel are obtained as functions of electric and rheologic parameters.

One of the rheologic models of a continuous medium which satisfactorily describes the flow of a finely-dispersed medium is the model of a viscoplastic fluid (Shvedov–Bingham plastic) [1-4] for which, during one-dimensional steady flow, the relation between shear stress and rate of deformation (rheologic law) takes the form

$$\tau=\tau_0\operatorname{sign}\left(\frac{du}{dy}\right)+\mu\frac{du}{dy},\quad |\tau|>\tau_0;$$
$$\frac{du}{dy}=0,\quad |\tau|<\tau_0. \qquad (1)$$

Here $\tau$ is the shear stress, $\tau_0$ and $\mu$ are, respectively, the maximum shear stress and the coefficient of dynamic viscosity (rheologic constants of the medium), u is the velocity of the medium, and y is the transverse coordinate.

Steady gradient flow of a viscoplastic fluid in a plane channel has been investigated [5]. In particular cases, a finely divided suspension may be formed by electrically charged particles. The presence of a volume density of electric charges in an external electric field leads to the realization of qualitatively new patterns of flow. Electrodynamic flow of a Newtonian fluid between parallel plates has been discussed [6]. A similar problem is considered below for a viscoplastic substance.

Let a viscoplastic fluid with an electric charge volume density $\rho=\rho(y)$ move in an external longitudinal homogeneous electric field of intensity $E_0$ under the influence of a time-invariant pressure gradient $P = -\partial p/\partial x$ (x is the longitudinal coordinate) between plane walls at $y = \pm L$ with the quantity

$$q_0=\int_{-L}^{L}\varrho(y)\,dy$$

being a known parameter of the problem. The distributions of electric charge density and of the induced transverse electric field $e=e(y)$ in a channel with dielectric walls are determined by the laws of electrodynamics and the condition that the transverse component of the current density goes to zero. Furthermore, the contributions from both the diffusive component of the current density and the mobility of electric charges are taken into consideration. If one takes as characteristic quantities L, $\rho_0 = q_0/L$, and $e_0 = q_0/\varepsilon\varepsilon_0$, where $\varepsilon$ is the dielectric constant of the medium and $\varepsilon_0$ is the dielectric constant of vacuum, then for the dimensionless distributions $\rho(y)$ and $e(y)$ there occur the following relations [6]:

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$$\varrho(y)=\frac{\alpha}{2\,\mathrm{tg}\,\alpha}\frac{1}{\cos^2\alpha y}; \qquad e(y)=\frac{1}{2}\frac{\mathrm{tg}\,\alpha y}{\mathrm{tg}\,\alpha}, \tag{2}$$

where the quantity $\alpha<\pi/2$ is defined as the root of the transcendental equation

$$\alpha\,\mathrm{tg}\,\alpha=bq_0L/4D\varepsilon\varepsilon_0.$$

Here b is the mobility and D the diffusion coefficient for the electric charges in the medium.

The equation of motion for an arbitrary electrically charged continuous medium in a plane channel has the form

$$P+\varepsilon\varepsilon_0E_0\frac{de}{dy}+\frac{d\tau}{dy}=0.$$

Transforming to dimensionless quantities, it is easy to obtain

$$\frac{d\tau}{dy}=-1-\Pi e\,\varrho(y), \tag{3}$$

where the quantity PL is taken as a characteristic quantity for stress, and the parameter for electric effect $\Pi e=q_0E_0/PL$ characterizes the relation between electric forces and pressure forces. The sign of $\Pi e$ is determined by the direction of the external electric field ($\Pi e \gtrless 0$ for $E_0 \gtrless 0$).

Integrating Eq. (3) and allowing for the flow symmetry condition $\tau(0)=0$, it is easy to obtain

$$\tau=-y-\frac{\Pi e}{2}\frac{\mathrm{tg}\,\alpha y}{\mathrm{tg}\,\alpha}. \tag{4}$$

Distribution curves for the shear stress $\tau=\tau(y)$ when $\alpha=1$ are shown in Fig. 1 as functions of the parameter $\Pi e$. Analytic investigation of Eq. (4) shows that for $\Pi e>\Pi_1=-\sin 2\alpha/\alpha$ and for $\Pi e<\Pi_2=-2\tan\alpha/\alpha$, the distribution of shear stress has no extremum in the range $-1<y<0$ (lower half of the channel). When the inequality $\Pi_2<\Pi e<\Pi_1$ is satisfied, Eq. (4) takes on an extremum value

$$\tau_*=\frac{1}{\alpha}\left(\arccos\sqrt{\frac{\Pi e}{\Pi_2}}-\frac{\Pi e}{\Pi_2}\sqrt{\frac{\Pi_2}{\Pi e}-1}\right)$$

at the point

$$y_*=-\frac{1}{\alpha}\arccos\sqrt{\frac{\Pi e}{\Pi_2}}.$$

When $\Pi e=\Pi_0=-2$, the stress at the channel wall goes to zero and decreases with further decrease in $\Pi e$. When $\Pi e=\Pi\omega$, where $\Pi\omega$ is defined as the root of the transcendental equation

$$-\frac{1}{\alpha}\left(\arccos\sqrt{\frac{\Pi e}{\Pi_2}}-\frac{\Pi e}{\Pi_2}\sqrt{\frac{\Pi_2}{\Pi e}-1}\right)=1+\frac{\Pi e}{2},$$

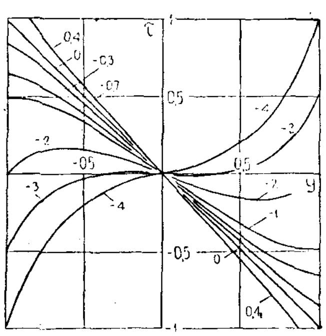


Fig. 1. Shear stress distribution for $\alpha=1$. Numbers on the curves are values of the parameter $\Pi e$.

the extremal value of the shear stress equals the absolute magnitude of the latter at the channel wall.

Returning to consideration of the features of viscoplastic flow in a medium, we rewrite the rheologic law (1) in the dimensionless form

$$\tau=S\,\mathrm{sign}\left(\frac{du}{dy}\right)+\frac{du}{dy}, \qquad |\tau|>S;$$
$$\frac{du}{dy}=0, \qquad |\tau|<S, \tag{5}$$

where $S=\tau_0/PL$ and the quantity $PL^2/\mu$ is a characteristic velocity. To construct velocity profiles for the medium, it is necessary to find solutions of the equation

$$|\tau(y)|=S, \tag{6}$$

which are boundaries between zones of viscous flow and zones of quasisolid motion. The following cases are then possible.

1. $\Pi e > \Pi_1$. There is a unique solution of Eq. (6) $y = y_0$ for all cases where $0 < S < 1 + {}^1\!/_2\Pi e$; in the boundary region $-1 < y < y_0$, there occurs viscous flow

$$u=\frac{1-y^2}{2}-\frac{1}{\alpha^2}\frac{\Pi e}{\Pi_2}\ln\frac{\cos\alpha y}{\cos\alpha}-S(1+y), \tag{7}$$

and in the center of the channel $y_0 < y < 0$, the medium moves as a unit at the velocity $u_0 = u(y_0)$. For cases where $S > 1 + {}^1\!/_2\Pi e$, there is no flow (channel-blocking effect).

2. $\Pi_0 \le \Pi_e < \Pi_1$. Flow does not occur in a viscoplastic medium for $S > \tau_*$. For $0 < S < \tau_*$, there are two solutions of Eqs. (6): $y = y_{01}$ and $y = y_{02}$. For definiteness, let $y_{01} > y_{02}$. Then a quasisolid zone adhering to the channel wall is formed in the region $-1 < y < y_{02}$; there is viscous flow in the region $y_{02} < y < y_{01}$, and a moving quasisolid core in the central region $y_{01} < y < 0$. The velocity distribution of the medium in the channel takes the form

$$u(y)=\begin{cases}0 \quad \text{for} \quad -1<y<y_{02},\\ \dfrac{y_{02}^2-y^2}{2}-\dfrac{1}{\alpha^2}\dfrac{\Pi e}{\Pi_2}\ln\dfrac{\cos\alpha y}{\cos\alpha y_{02}}+S(y_{02}-y) \quad \text{for} \quad y_{02}<y<y_{01},\\ \dfrac{y_{02}^2-y_{01}^2}{2}-\dfrac{1}{\alpha^2}\dfrac{\Pi e}{\Pi_2}\ln\dfrac{\cos\alpha y_{01}}{\cos\alpha y_{02}}+S(y_{02}-y_{01}) \quad \text{for} \quad y_{01}<y<0.\end{cases} \tag{8}$$

3. $\Pi_W \le \Pi e < \Pi_0$. In this range of the parameter $\Pi e$, the extremal stress value $\tau_*$ is greater than the absolute magnitude of the shear stress at the channel wall. In this case, channel blocking occurs for $S > \tau_*$; when $|1 + {}^1\!/_2\Pi e| < S < \tau_*$, Eq. (6) has two roots and the flow pattern is similar to that discussed in paragraph 2. When $0 < S < |1 + {}^1\!/_2\Pi e|$, Eq. (6) has three roots: $y_{01}$, $y_{02}$, and $y_{03}$ (in decreasing order). In this case, two zones of quasisolid motion and two zones of viscous flow occur. It should be noted that $du/dy < 0$ in the boundary zone of viscous flow and $du/dy > 0$ in the central viscous zone. Considering this, it is easy to describe the velocity distribution for the medium in the channel:

$$u(y)=\begin{cases}\dfrac{1-y^2}{2}-\dfrac{1}{\alpha^2}\dfrac{\Pi e}{\Pi_2}\ln\dfrac{\cos\alpha y}{\cos\alpha}+S(1+y) \quad \text{for} \quad -1<y<y_{03},\\ \dfrac{1-y_{03}^2}{2}-\dfrac{1}{\alpha^2}\dfrac{\Pi e}{\Pi_2}\ln\dfrac{\cos\alpha y_{03}}{\cos\alpha}+S(1+y_{03}) \quad \text{for} \quad y_{03}<y<y_{02},\\ \dfrac{1-y_{03}^2}{2}-\dfrac{1}{\alpha^2}\dfrac{\Pi e}{\Pi_2}\ln\dfrac{\cos\alpha y_{03}}{\cos\alpha}+S(1+y_{03})+\dfrac{y_{02}^2-y^2}{2}-\\ \quad -\dfrac{1}{\alpha^2}\dfrac{\Pi e}{\Pi_2}\ln\dfrac{\cos\alpha y}{\cos\alpha y_{02}}+S(y_{02}-y) \quad \text{for} \quad y_{02}<y<y_{01},\\ \dfrac{1-y_{03}^2}{2}-\dfrac{1}{\alpha^2}\dfrac{\Pi e}{\Pi_2}\ln\dfrac{\cos\alpha y_{03}}{\cos\alpha}+S(1+y_{03})+\dfrac{y_{02}^2-y_{01}^2}{2}-\\ \quad -\dfrac{1}{\alpha^2}\dfrac{\Pi e}{\Pi_2}\ln\dfrac{\cos\alpha y_{01}}{\cos\alpha y_{02}}+S(y_{02}-y_{01}) \quad \text{for} \quad y_{01}<y<0.\end{cases} \tag{9}$$

4. $\Pi e < \Pi_W$. In this mode, the absolute value of the shear stress at the wall exceeds the extremum $\tau_*$. When $0 < S < \tau_*$, Eq. (6) obviously has three solutions and the flow pattern is described by Eqs. (9). When $\tau_* < S < |1 + {}^1\!/_2\Pi e|$, Eq. (6) has the single solution $y = y_0$, which corresponds to the formation of a single quasisolid zone in the center of the channel and also a zone of viscous flow with $du/dy < 0$ in the boundary region. The distribution of medium velocity in the channel takes the form

$$u(y)=\begin{cases}\dfrac{1-y^2}{2}-\dfrac{1}{\alpha^2}\dfrac{\Pi e}{\Pi_2}\ln\dfrac{\cos\alpha y}{\cos\alpha}+S(1+y) \quad \text{for} \quad -1<y<y_0,\\ \dfrac{1-y_0^2}{2}-\dfrac{1}{\alpha^2}\dfrac{\Pi e}{\Pi_2}\ln\dfrac{\cos\alpha y_0}{\cos\alpha}+S(1+y_0) \quad \text{for} \quad y_0<y<0.\end{cases} \tag{10}$$

When $S > |1 + {}^1\!/_2\Pi e|$, the channel-blocking effect occurs.

In the equations obtained, one can perform two limit transitions, $S \to 0$ and $\Pi e \to 0$. The first, which

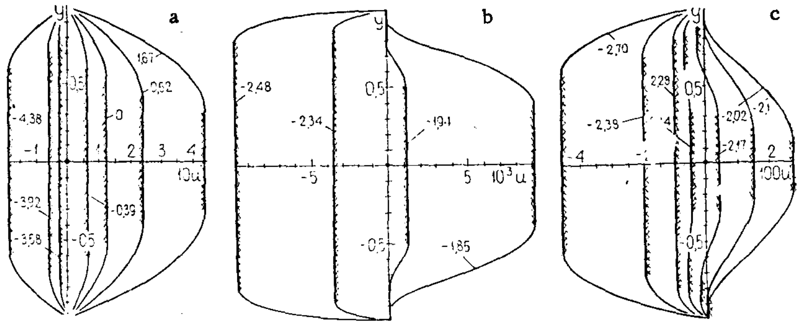

Fig. 2. Distribution of medium velocities in the channel as a function of the parameter Пе $\alpha = \pi/4$. a) S = 0.5; b) S = 0.1; c) S = 0.03.

corresponds to the transition to a Newtonian fluid, leads to the relations in [6]. The second limiting transition corresponds to gradient flow of an uncharged fluid [5].

The analysis of possible flow modes in a channel given for a charged viscoplastic fluid was carried out for particular assigned values of the parameter Пе while the plasticity parameter S was varied. A similar study can be carried out for specific values of S as a function of the quantity Пе. Results of such calculations are shown in Fig. 2. Depending on the value of the plasticity parameter, it is easy to see that when Пе changes from positive to negative values it is possible to have reversal of flow through channel blocking without the creation of a new zone of quasisolid motion (Fig. 2a), with the formation of a quasisolid zone adhering to the wall (Fig. 2b), and also with the formation of two viscous and two quasisolid zones (Fig. 2c). For better visualization, the zones of quasisolid motion in Fig. 2 are emphasized by additional hatching.

Gradient steady-state flow in a plane channel for an electrically conducting viscoplastic fluid in a uniform magnetic field was investigated earlier [7]. It is interesting to note that where the effect of the transverse magnetic field reduces merely to a broadening of the quasisolid core; the superposition of an electric field parallel to the flow velocity on the motion of a viscoplastic fluid with a volume electric charge density qualitatively changes the pattern of the velocity distribution in the medium.

We point out that with electrohydrodynamic flow in a plane channel, the volume electromagnetic force is independent of flow velocity (in contrast to magnetohydrodynamic flow) and therefore Eq. (3) for shear stress can be integrated without involving the rheologic law. Therefore, the distribution of shear stress in the channel given by Eq. (4) is valid for a medium under any rheologic law. The specific form of the rheologic law is only important in calculating the velocity distribution in the channel.

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